MATHEMATICAL METHODS
(MATH 2001)

Time Allotted : 2½ hrs **Full Marks : 60**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 4 (four) from Group B to E, taking one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A

1. Answer any twelve: **12 × 1 = 12**

Choose the correct alternative for the following

(i) The period of $cos(2\pi x)$ is
(a) 2π (b) 1 (c) 2 (d) π

(ii) If the Fourier transform of $f(x)$ be $F(s)$, then Fourier inverse transform of $F(s-a)$ is
(a) e^{-iax} (b) e^{iax} (c) $e^{-iax}f(x)$ (d) $e^{ias}f(x)$.

(iii) The one dimensional wave equation is: (where c is treated as constant)
(a) $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ (b) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$
(c) $\left(\frac{\partial u}{\partial t}\right)^2 = c^2 \frac{\partial^2 u}{\partial x^2}$ (d) $\frac{\partial^2 u}{\partial t^2} = c^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$.

(iv) The solution of the partial differential equation $xp + yq = x$ is
(a) $\Phi\left(\frac{x^2}{2} - y, y - z^2\right) = 0$ (b) $\Phi(x + y, x + z) = 0$
(c) $\Phi(x^2 + y^2, y - z) = 0$ (d) $\Phi(x - y, x - z) = 0$

(v) If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ in $-\pi \leq x \leq \pi$ is an even function, then which of the following is always zero?
(a) b_n (b) a_n (c) a_0 (d) both a_n and b_n.

(vi) The value of p for which the function $2x - x^2 + py^2$ will be harmonic is
(a) 0 (b) 1 (c) 2 (d) 3

(vii) The value of the integral $\int_C \frac{dz}{z+3}$, where C is the circle $|z| = 1$, is
(a) 0 (b) 1 (c) 2 (d) -1.

(viii) Let $f(z) = \frac{z^2}{(z-1)^2(z+3)}$, then $z = 1$ is a pole of order
(a) 0 (b) 1 (c) 2 (d) 3.

(ix) The regular singular point(s) of the ordinary differential equation $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0$ is/are

(a) $1, 0$ (b) only 1 (c) $1, -1$ (d) only -1.

(x) The value of $P_n(-1)$ is

(a) -1 (b) 1 (c) $(-1)^n$ (d) 0.

Fill in the blanks with the correct word

(xi) The period of the function $2\cos^2 x + \sin x - 4\cos 2x$ is ________.

(xii) The Fourier sine transform of the function $f(x) = \frac{1}{x}$ is ________.

(xiii) The order and degree of the partial differential equation $\left(\frac{\partial^3 f}{\partial x^3}\right)^2 + 2\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial y} = 0$ is ________.

(xiv) The value of the Bessel's function $J_{-\frac{1}{2}}(x)$ is ________.

(xv) The value of $\lim_{z\to -i} \frac{z^2+1}{z+i}$ is ________.

Group - B

2. (a) Show that $u(x,y) = 4xy - x^3 + 3xy^2$ is a harmonic function. Hence, obtain its harmonic conjugate function $v(x,y)$, so that $f = u + iv$ is an analytic function.

[(MATH2001.1, MATH2001.2)(Apply/IOCQ)]

(b) Determine the poles and the residue at each pole of the function $f(z) = \frac{z^2}{(z-1)^2(z+2)}$.

[(MATH2001.1,MATH2001.2)(Understand/LOCQ)]

6 + 6 = 12

3. (a) Evaluate using Cauchy's integral formula $\oint_C \frac{e^{2z}}{(z+1)^4} dz$, where C is the circle $|z| = 3$.

[(MATH2001.1, MATH2001.2)(Apply/IOCQ)]

(b) Show that the function $f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ satisfies Cauchy-Riemann equations at the origin but $f'(0)$ does not exist.

[(MATH2001.1,MATH2001.2)(Analyse/HOCQ)]

6 + 6 = 12

Group - C

4. (a) Find the Fourier series expansion of the following function:

$$f(x) = \begin{cases} -x, & -2 < x \leq 0 \\ x, & 0 < x < 2 \end{cases}$$

[(MATH2001.1,MATH2001.3,MATH2001.4)(Apply/IOCQ)]

(b) Find the Fourier inverse transform of the function $F(s) = \frac{1}{s^2+4s+13}$.

[(MATH2001.1, MATH2001.3, MATH2001.4)(Remember/LOCQ)]

7 + 5 = 12

5. (a) Obtain the half range Fourier cosine series of $f(x) = x$, in $0 < x < 2$.

[(MATH2001.3, MATH2001.4)(Understand/LOCQ)]

(b) Find the Fourier transform of the function $f(x) = \begin{cases} 1, |x| \leq 1 \\ 0, |x| > 1 \end{cases}$.

Hence show that $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.

[(MATH2001.3,MATH2001.4)(Analyse/IOCQ)]

5 + 7 = 12

Group - D

6. (a) Define generating function for Bessel's function and hence show that $2J_n{}'(x) = J_{n-1}(x) - J_{n+1}(x)$.

[(MATH2001.5)(Apply/IOCQ)]

(b) Prove that $(2n+1)x\, P_n(x) = (n+1)\, P_{n+1}(x) + n\, P_{n-1}(x)$, where $P_n(x)$ denotes the Legendre polynomial of degree n.

[(MATH2001.5)(Apply/IOCQ)]

6 + 6 = 12

7. (a) Find the series solution of the following differential equation:

$(1 - x^2)\frac{d^2y}{dx^2} + 2y = 0$ about $x = 0$.

[(MATH2001.5)(Evaluate/HOCQ)]

(b) Using Rodrigue's formula find $P_0(x)$, $P_1(x)$ and $P_2(x)$ where $P_n(x)$ denotes the Legendre polynomial of degree n.

[(MATH2001.5)(Remember/LOCQ)]

8 + 4 = 12

Group - E

8. (a) Form a partial differential equation from the relation $\Phi(x^2 + y^2, z - xy) = 0$, by eliminating the arbitrary function.

[(MATH2001.1,MATH2001.6)(Understand/LOCQ)]

(b) Apply Lagrange's method to solve the following linear partial differential equation: $(mz - ny)p + (nx - lz)q = (ly - mx)$, where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$.

[(MATH2001.1,MATH2001.6)(Apply/IOCQ)]

6 + 6 = 12

9. (a) Find the general solution of the following partial differential equation:

$\frac{\partial^2 z}{\partial x^2} - 7\frac{\partial^2 z}{\partial x \partial y} + 12\frac{\partial^2 z}{\partial y^2} = e^{x-y}$.

[(MATH2001.6)(Apply/IOCQ)]

(b) A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3\left(\frac{\pi x}{l}\right)^3$. If it is released from rest from this position, find the displacement $y(x,t)$.

[(MATH2001.6)(Create/HOCQ)]

5 + 7 = 12

Cognition Level	LOCQ	IOCQ	HOCQ
Percentage distribution	27.08	51.04	21.88

Course Outcome (CO):

After the completion of the course students will be able to

MATH2001.1	Construct appropriate mathematical models of physical systems.
MATH2001.2	Recognize the concepts of complex integration, Poles and Residuals in the stability analysis of engineering problems.
MATH2001.3	Generate the complex exponential Fourier series of a function and make out how the complex Fourier coefficients are related to the Fourier cosine and sine coefficients.
MATH2001.4	Interpret the nature of a physical phenomena when the domain is shifted by Fourier Transform e.g. continuous time signals and systems.
MATH2001.5	Develop computational understanding of second order differential equations with analytic coefficients along with Bessel and Legendre differential equations with their corresponding recurrence relations.
MATH2001.6	Master how partial differentials equations can serve as models for physical processes such as vibrations, heat transfer etc.

**LOCQ: Lower Order Cognitive Question; IOCQ: Intermediate Order Cognitive Question; HOCQ: Higher Order Cognitive Question.*