

MATHEMATICAL METHODS (MTH2001)

Time Allotted : 2½ hrs

Full Marks : 60

Figures out of the right margin indicate full marks.

*Candidates are required to answer Group A and
any 4 (four) from Group B to E, taking one from each group.*

Candidates are required to give answer in their own words as far as practicable.

Group – A

1. Answer any twelve:

12 × 1 = 12

Choose the correct alternative for the following

- (i) For an infinite power series $\sum_{n=0}^{\infty} c_n x^n$ if $\lim_{n \rightarrow \infty} \left| \frac{c_n}{c_{n+1}} \right| = R$ then the radius of convergence is
 (a) $1/R$ (b) R (c) 1 (d) ∞
- (ii) For a differential equation with a regular singular point at $x = 0$, the indicial equation is obtained by:
 (a) Substituting the power series into the differential equation and setting the lowest power of x to zero.
 (b) Differentiating the power series twice.
 (c) Integrating the power series.
 (d) Finding the roots of the characteristic polynomial.
- (iii) Which of the following is the correct expression for the generating function of Legendre polynomials?
 (a) $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$ (b) $\frac{1}{1-xt} = \sum_{n=0}^{\infty} P_n(x)t^n$
 (c) $\frac{1}{(1-xt)^{1/2}} = \sum_{n=0}^{\infty} P_n(x)t^n$ (d) $\frac{1}{\sqrt{1+x^2t}} = \sum_{n=0}^{\infty} P_n(x)t^n$
- (iv) What is the value of the second Legendre polynomial $P_2(x)$
 (a) $\frac{1}{2}(3x^2 - 1)$. (b) $2x^2 - 1$.
 (c) $\frac{1}{2}(5x^2 - 3)$. (d) $3x^2 - 1$.
- (v) Which of the following properties is true for Hermite polynomials?
 (a) $H_n(-x) = (-1)^n H_n(x)$ (b) $H_n(-x) = H_n(x)$
 (c) $H_n(-x) = (-1)^{n+1} H_n(x)$ (d) $H_n(-x) = -H_n(x)$
- (vi) The partial differential equation corresponding to $z = (x + it) + g(x - it)$ after eliminating the arbitrary function is
 (a) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial t} = 0$ (b) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$
 (c) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = -1$ (d) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 1$

- (vii) Which of the following is true for a homogeneous PDE?
 (a) It contains no derivatives.
 (b) It has no boundary conditions.
 (c) The PDE has all terms with the dependent variable and its derivatives.
 (d) The PDE has the right-hand side equal to zero.
- (viii) The heat equation in one dimension is given by:
 (a) $u_t = c^2 u_{xx}$. (b) $u_t + c^2 u_{xx} = 0$.
 (c) $u_x = c^2 u_{tt}$. (d) $u_x + cu_t = 0$.
- (ix) Complementary function of the given PDE $(D^2 - 7DD' + 6D'^2)z = 0$ is given by:
 (a) $\phi_1(y+x) + \phi_2(y-6x)$. (b) $\phi_1(y+x) + \phi_2(y+6x)$.
 (c) $\phi_1(y+x) \cdot \phi_2(y+6x)$. (d) $\phi_1(y-x) + \phi_2(y-6x)$.
- (x) Which of the following is a major advantage of using the Laplace transform in solving boundary value problems?
 (a) It converts a partial differential equation into a linear algebraic equation.
 (b) It converts a partial differential equation into an ordinary differential equation.
 (c) It automatically provides the exact solution to the differential equation.
 (d) It eliminates the need for boundary conditions.

Fill in the blanks with the correct word

- (xi) If the difference between the roots of the indicial equation is an integer, the second solution may involve a _____ term.
- (xii) The generating function for Hermite polynomials $H_n(x)$ is _____.
- (xiii) In Lagrange's method, the characteristic equations are written as $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, and their solutions are two independent _____ of these equations.
- (xiv) The general solution in Lagrange's method is expressed as a function of the two independent integrals, $f(u, v) = 0$, where u and v are the _____.
- (xv) The general solution of the one-dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ is given by _____.

Group - B

2. (a) The equation $x'' + e^t x = 0$ has a series solution $x(t) = \sum_{k=0}^{\infty} a_k t^k$ which satisfies $x(0) = 1$ and $x'(0) = 0$. Determine the coefficients a_0, a_1, a_2, a_3 and a_4 .
 [(MTH2001.1, MTH2001.2)(Evaluate/HOCQ)]
- (b) For the equation $t^2 x'' + te^t x' + x = 0$, show that roots of the indicial equation are i and $-i$.
 [(MTH2001.1, MTH2001.2)(Analyse/IOCQ)]
- 6 + 6 = 12**
3. (a) Show that infinity is a regular singular point for
 $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$
 [(MTH2001.1, MTH2001.2)(Evaluate/HOCQ)]

(b) The equation

$x'' - 2t^2x' + 4tx = t^2 + t + 1$
has a series solution $\sum_{n=0}^{\infty} c_n t^n$ then find the recurrence relation.

[[MTH2001.1, MTH2001.2](Apply/IOCQ)]

5 + 7 = 12

Group - C

4. (a) Prove that $\lim_{z \rightarrow 0} \frac{J_n(z)}{z^n} = \frac{1}{2^n \Gamma(n+1)^n}$, where $n > -1$. [[MTH2001.3](Evaluate/HOCQ)]

(b) Express $x^4 + 2x^3 + 2x^2 - x - 3$ in terms of Hermite's polynomials. [[MTH2001.3](Evaluate/HOCQ)]

6 + 6 = 12

5. (a) Prove that $P'_n(1) = \frac{1}{2}n(n+1)$. [[MTH2001.3](Apply/IOCQ)]

(b) Evaluate $\int_{-\infty}^{\infty} x e^{-x^2} H_n(x) H_m(x) dx$. [[MTH2001.3](Evaluate/HOCQ)]

7 + 5 = 12

Group - D

6. (a) Solve the following partial differential equation by applying Lagrange's method:
 $(mz - ny)p + (nx - lz)q = ly - mx$, where l, m, n are constants and $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$. [[MTH2001.4, MTH2001.5](Understand/LOCQ)]

(b) Construct a partial differential equation by eliminating the arbitrary function f from the relation: $z = e^{ax+by} f(ax + by)$. [[MTH2001.4, MTH2001.5](Apply/IOCQ)]

6 + 6 = 12

7. (a) By a suitable substitution, solve the following non-linear partial differential equation $z^2(p^2 + q^2 + 1) = c^2$ where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, and c is a constant. [[MTH2001.4, MTH2001.5](Apply/IOCQ)]

(b) Solve the linear partial differential equation: $z(p - q) = z^2 + (x + y)^2$ where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$. [[MTH2001.4, MTH2001.5](Apply/IOCQ)]

6 + 6 = 12

Group - E

8. (a) A tightly stretched string with fixed end points $x = 0$ and $x = l$ is pulled from its middle portion and the initial position is given by $y = a \sin\left(\frac{\pi x}{l}\right)$. It is released from rest at this position. Prove that the displacement of any point at a distance x from one end at time t is given by

$y(x, t) = a \sin\left(\frac{\pi x}{l}\right) \cos\left(\frac{\pi ct}{l}\right)$, c is a constant [[MTH2001.5, MTH2001.6](Apply/IOCQ)]

(b) Solve the PDE $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = xy$. [[MTH2001.5, MTH2001.6](Remember/LOCQ)]

7 + 5 = 12

9. (a) Solve Laplace's equation in a rectangle:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0; \quad 0 < x < a, 0 < y < b;$$

with boundary conditions $u(0, y) = u(a, y) = 0, u(x, 0) = 0, u(x, b) = f(x)$.

[(MTH2001.5, MTH2001.6)(Evaluate/HOCQ)]

- (b) Solve the one-dimensional heat equation:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l$$

with the boundary conditions $u(0, t) = u(l, t) = 0$ and the initial condition $u(x, 0) = f(x)$.

[(MTH2001.5, MTH2001.6)(Analyse/IOCQ)]

6 + 6 = 12

Cognition Level	LOCQ	IOCQ	HOCQ
Percentage distribution	12.5	50	37.5