INTRODUCTION TO COMPLEX AND FOURIER ANALYSIS
(MTH2101)

Time Allotted : 2½ hrs **Full Marks : 60**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 4 (four) from Group B to E, taking one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A

1. Answer any twelve: **12 × 1 = 12**

Choose the correct alternative for the following

(i) If $3y - 5x^2 + my^2$ is a harmonic function, then the value of m is
(a) 2 (b) 3 (c) 5 (d) 0.

(ii) The function $f(z) = \bar{z}$ is
(a) continuous at $z = 0$ (b) differentiable at $z = 0$
(c) analytic at $z = 0$ (d) differentiable everywhere.

(iii) The value of $\oint_C \frac{z^2-z+1}{z-1} dz$, C being $|z| = \frac{e}{3}$ is
(a) $2\pi i$ (b) $\frac{1}{2\pi i}$ (c) 0 (d) πi.

(iv) The residue of $f(z) = \frac{2+3\sin \pi z}{z(z-1)^2}$ at $z = 0$ is
(a) 1 (b) 3 (c) 2 (d) 1.

(v) If $f(x) = x\sin x$, $-\pi < x \le \pi$ be expanded in a Fourier series as $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$, then a_1 is equal to
(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$.

(vi) The function $f(x) = \begin{cases} \cos 2x, & 0 < x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \le x < \pi \end{cases}$ is
(a) an odd function (b) an even function
(c) neither an odd nor an even function (d) both an odd and an even function.

(vii) Sum of two odd functions is
(a) an even function (b) an odd function
(c) neither an even nor an odd function (d) both an even and an odd function.

(viii) The Fourier cosine transform of e^{-x} is
(a) $\frac{s}{1+s^2}$ (b) $\frac{1}{1+s^2}$ (c) $\frac{1}{1-s^2}$ (d) $\frac{s}{1-s^2}$.

(ix) If $F\{f(x)\} = \bar{f}$, then $F\{f(x)\cos ax\}$ is equal to

(a) $\frac{1}{2}[\bar{f}(p+a) + \bar{f}(p-a)]$ (b) $\frac{1}{2}[\bar{f}(p-a) - \bar{f}(p+a)]$

(c) $\frac{1}{2}[\bar{f}(p+a) - \bar{f}(p-a)]$ (d) $\frac{1}{2}[\bar{f}(p-a) + \bar{f}(p+a)]$.

(x) If $F(s)$ be the Fourier transform of $f(t)$, then Fourier transform of $tf(t)$ is

(a) $\frac{d}{ds}\{F(s)\}$ (b) $i\frac{d}{ds}\{F(s)\}$

(c) $-i\frac{d}{ds}\{F(s)\}$ (d) $\frac{d}{ds}\{sF(s)\}$.

Fill in the blanks with the correct word

(xi) The value of $\lim_{z\to i}\frac{iz+1}{z-i}$ is ________________.

(xii) The residue of $\frac{z^2}{z^2+a^2}$ at $z = ia$ is ________________.

(xiii) If $f(x) = |x|, -\pi < x < \pi$ be presented in Fourier series as $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos\ nx + b_n\sin\ nx)$, then the value of a_0 is ________________.

(xiv) The period of the function $f(x) = |\sin x|$ is ________________.

(xv) The Fourier transform of $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ is ________________.

Group - B

2. (a) For the function $f(z) = \sqrt{|xy|}$, show that the Cauchy-Riemann equations are satisfied at (0,0), but the function is not differentiable at that point.

[(MTH2101.1)(Apply/IOCQ)]

(b) Consider the Riemann sphere given by $x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$. If $(1, 0)$ be a point on the complex plane, then find its stereographic projection.

[(MTH2101.1)(Remember/LOCQ)]

(c) Show that $f(z) = \bar{z}$ is nowhere analytic. *[(MTH2101.1)(Understand/LOCQ)]*

6 + 2 + 4 = 12

3. (a) Find the image of the circle $|w| = 1$ under the transformation $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$.

[(MTH2101.1)(Apply/IOCQ)]

(b) Show that the function $f(z) = xy + iy$ is everywhere continuous but is not analytic at any point. *[(MTH2101.1)(Remember/LOCQ)]*

6 + 6 = 12

Group - C

4. (a) Use Cauchy's Integral formula to evaluate $\oint_C \frac{\cos^3 z}{\left(z-\frac{\pi}{4}\right)^3}dz$, where C is the circle $|z| = 1$. *[(MTH2101.2,MTH2101.3)(Remember/LOCQ)]*

(b) State the Residue theorem. Use it to evaluate $\oint_C \frac{z}{(z-1)(z-2)^2} dz$, where C is $|z-2| = \frac{1}{2}$. *[(MTH2101.2, MTH2101.3)(Apply/IOCQ)]*

6 + 6 = 12

5. (a) Find Laurent series about the indicated singularity for the function $f(z) = (z+2)\sin\left(\frac{1}{z-2}\right)$ at $z = 2$. *[(MTH2101.2,MTH2101.3)(Evaluate/HOCQ)]*

(b) Evaluate $\oint_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz$, if C is the circle $|z| = 1$. *[(MTH2101.2,MTH2101.3)(Remember/LOCQ)]*

6 + 6 = 12

Group - D

6. (a) Find the Fourier series of the function $f(x) = \begin{cases} 0, & \text{for } -\pi < x \le 0 \\ \frac{\pi x}{4}, & \text{for } 0 < x < \pi \end{cases}$ in $[-\pi, \pi]$. *[(MTH2101.4)(Remember/LOCQ)]*

(b) Find the half range cosine series of $(x) = \begin{cases} \pi x, & 0 < x < 1 \\ \pi(2-x), & 1 \le x < 2 \end{cases}$.
Hence using Parseval's identity show that $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$. *[(MTH2101.4)(Evaluate/HOCQ)]*

5 + 7 = 12

7. (a) Find the Fourier series expansion of the function $f(x) = |\sin x|$ in $-\pi \le x \le \pi$. *[(MTH2101.4)(Analyse/IOCQ)]*

(b) Determine the radius of convergence of the power series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$. *[(MTH2101.4)(Understand/LOCQ)]*

7 + 5 = 12

Group - E

8. (a) Find the Fourier transform of $f(x) = \begin{cases} 1 - x^2, & |x| < 1 \\ 0, & |x| \ge 1 \end{cases}$.
Use it to evaluate $\int_0^{\infty} \frac{x\cos x - \sin x}{x^3} \cos\frac{x}{2} dx$. *[(MTH2101.5,MTH2101.6)(Evaluate/HOCQ)]*

(b) Find the Fourier cosine transform of $f(x) = \sin x$ in $0 < x < \pi$. *[(MTH2101.5,MTH2101.6)(Remember/LOCQ)]*

7 + 5 = 12

9. (a) Find the Fourier transform of $f(x) = \frac{\sin ax}{x}$.
Hence prove using Parseval's Identity $\int_{-\infty}^{\infty} \frac{\sin^2 ax}{x^2} dx = 4a\pi$. *[(MTH2101.5,MTH2101.6)(Evaluate/HOCQ)]*

(b) Obtain the Fourier sine transform of $f(x) = \frac{1}{x}$. *[(MTH2101.5,MTH2101.6)(Remember/LOCQ)]*

7 + 5 = 12

Cognition Level	LOCQ	IOCQ	HOCQ
Percentage distribution	45.83	26.04	28.13