# LINEAR ALGEBRA
## (MATH 4126)

**Time Allotted : 2½ hrs** **Full Marks : 60**

***Figures out of the right margin indicate full marks.***

***Candidates are required to answer Group A and any 4 (four) from Group B to E, taking one from each group.***

***Candidates are required to give answer in their own words as far as practicable.***

## Group – A

1. Answer any twelve: **12 × 1 = 12**

*Choose the correct alternative for the following*

(i) If $A$ be a singular matrix then an eigenvalue of $A$ is
(a) 1 (b) 2 (c) $-1$ (d) 0.

(ii) If $A$ is a $3 \times 3$ diagonalizable matrix, then the number of linearly independent eigenvectors of $A$ is
(a) 1 (b) 2 (c) 3 (d) 4.

(iii) The quadratic form $2x^2 + 3y^2 + 2z^2 - 2xy - 2yz$ is
(a) negative definite (b) indefinite
(c) positive definite (d) positive semi-definite.

(iv) Let $V$ be a vector space over the field $F$. Let $\theta$ be the zero vector of $V$, 0 be the zero of $F$ and $\alpha \in F$, then
(a) $\alpha.\theta > \theta$ (b) $\alpha.\theta < \theta$ (c) $\alpha.\theta = \theta$ (d) $\alpha.\theta = 0$.

(v) The dimension of the vector space spanned by $(-3, 0, 1)$, $(1, 2, 1)$ and $(3, 0, -1)$ is
(a) 1 (b) 2 (c) 3 (d) 0.

(vi) The set $V = \{(x, y) \in \mathbb{R}^2 : xy \geq 0\}$ is
(a) a vector space over $\mathbb{R}^2$ (b) a vector space over $\mathbb{R}$
(c) not a vector space over $\mathbb{R}$ (d) none of the above.

(vii) The norm of $u = (-1, 2, 3)$ in $\mathbb{R}^3$ with standard inner product is
(a) 14 (b) $-14$ (c) $\sqrt{14}$ (d) $-6$.

(viii) Consider the vectors $u = (1, 1, 1)$ and $v = (1, 2, -3)$ in $\mathbb{R}^3$. The value of $\langle u, v \rangle$ is
(a) 0 (b) $-1$ (c) 2 (d) 14.

(ix) Let $T: \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation and the dimension of $Ker\ T$ is 2. Then the dimension of $Im\ T$ is
(a) 1 (b) 2 (c) 3 (d) 4.

(x) A linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T(x,y) = (x+y, x)\ \forall (x,y) \in \mathbb{R}^2$. The nullity of $T$ is

(a) 3 (b) 2 (c) 1 (d) 0.

*Fill in the blanks with the correct word*

(xi) If $5x_1^2 + 2x_1x_2 - x_2^2$ is a real quadratic form in two variables $x_1$ and $x_2$, then the associated matrix is ________________.

(xii) The value of $x$ for which the set of vectors $\{(1,2,1), (x,3,1), (2,x,0)\}$ are linearly independent in $\mathbb{R}^3$ is ________________.

(xiii) The set containing the zero vector is linearly ______________.

(xiv) If $u = (1,3,-4,2) \in \mathbb{R}^4$, then $\|u\|$ is ____________.

(xv) In a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ if $T\{(1,0),(0,1)\} = \{(2,7),(1,3)\}$, then the matrix representation of $T$ with respect to the standard basis of $\mathbb{R}^2$ is ________.

## Group - B

2. (a) Find the Singular Value Decomposition of the matrix $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$.

*[(MATH4126.1, MATH4126.6)(Evaluate/HOCQ)]*

(b) If $\lambda$ is an eigenvalue of a square matrix $A$, then show that $\lambda^2$ is an eigenvalue of $A^2$.

*[(MATH4126.1, MATH4126.6)(Analyse/IOCQ)]*

**8 + 4 = 12**

3. (a) Show that the matrix $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is diagonalizable and also find the diagonal form.

*[(MATH4126.1, MATH4126.6)(Apply/IOCQ)]*

(b) Reduce the quadratic form $5x^2 + y^2 + 10z^2 - 4yz - 10zx$ to the normal form and show that it is positive definite.

*[(MATH4126.1, MATH4126.6)(Analyse/IOCQ)]*

**6 + 6 = 12**

## Group - C

4. (a) Determine whether the set of vectors $\{(2,-1,1), (2,0,3), (1,1,-2)\}$ forms a basis of the vector space $\mathbb{R}^3$ or not.

*[(MATH4126.2)(Understand/LOCQ)]*

(b) Find the values of $k$ so that the vectors $(1,-1,2)$, $(0,k,3)$ and $(-1,2,3)$ are linearly independent.

*[(MATH4126.2)(Remember/LOCQ)]*

(c) Show that the subset $S = \{(x,y,z) \in \mathbb{R}^3 : 3x - y + z = 0\}$ of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$. Hence, find a basis and dimension of $S$.

*[(MATH4126.2)(Apply/IOCQ)]*

**4 + 2 + 6 = 12**

5. (a) Let $V$ be the vector space $\mathbb{R}^3$ over the field of all real numbers and $W_1 = \{(0,y,z): y,z \in \mathbb{R}\}$ and $W_2 = \{(x,y,0): x,y \in \mathbb{R}\}$.

(i) Find $W_1 \cap W_2$. Is it a subspace of $V$? Justify your answer.

(ii) Find $W_1 \cup W_2$. Is it a subspace of $V$? Justify your answer.

*[(MATH4126.2)(Understanding/HOCQ)]*

(b) Find the conditions on $(x, y, z)$ such that it belongs to the span of the vectors $(2,1,0), (1,-1,2)$ and $(0,3,-4)$. *[(MATH4126.2)(Apply/IOCQ)]*

**(3 + 3) + 6 = 12**

## Group - D

6. (a) State and prove the Pythagoras theorem for norms of vectors in an inner product space. *[(MATH4126.3. MATH4126.4)(Remember/LOCQ)]*

(b) If $u$ and $v$ be two vectors in a real inner product space and $\|u\| = \|v\|$, then show that $\langle u+v, u-v\rangle = 0$. *[(MATH4126.3. MATH4126.4)(Analyse/IOCQ)]*

(c) Let $\langle u, v\rangle$ be the standard inner product on $\mathbb{R}^2$. Let $\alpha = (1,2), \beta = (-1,1)$. If $\gamma$ is a vector such that $\langle \alpha, \gamma\rangle = -1$ and $\langle \beta, \gamma\rangle = 3$, then find the vector $\gamma$. *[(MATH4126.3. MATH4126.4)(Apply/IOCQ)]*

**3 + 3 + 6 = 12**

7. (a) If $V$ be a vector space of all polynomials in $t$ with inner product given by $\langle f, g\rangle = \int_0^1 f(t)g(t)dt$, where $f(t), g(t) \in V$. Now for $f(t) = 3t - 5$ and $g(t) = t^2$ find (i) $\langle f, g\rangle$, (ii) $\|f\|$ and (iii) $\|g\|$. *[(MATH4126.3. MATH4126.4)(Remember/LOCQ)]*

(b) Use Gram – Schmidt process to the vectors $(1,0,1), (1,1,1)$ and $(1,3,4)$ to obtain an orthogonal and corresponding orthonormal basis for $\mathbb{R}^3$ with the standard inner product. *[(MATH4126.3. MATH4126.4)(Evaluate/HOCQ)]*

**6 + 6 = 12**

## Group - E

8. (a) If a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x, y, z) = (x + 2y + 3z, 3x + 2y + 5z, x + y + 2z), \forall\, (x, y, z) \in \mathbb{R}^3$, then find the rank of $T$ and nullity of $T$. Hence verify the Rank-Nullity theorem. *[(MATH4126.5)(Analyse/IOCQ)]*

(b) Consider the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (3x + 4y, 2x - 5y)$ and the basis of $\mathbb{R}^2$: $S = \{(1,2), (2,3)\}$. Find the matrix representing the linear transformation $T$ relative to the basis $S$. *[(MATH4126.5)(Evaluate/HOCQ)]*

**6 + 6 = 12**

9. (a) Show that the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x - y, x + 2y, y + 3z), \forall (x, y, z) \in \mathbb{R}^3$ is one-to-one and onto. *[(MATH4126.5)(Analyse/IOCQ)]*

(b) The matrix of a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ relative to the ordered basis $\{(-1,1,1), (1,-1,1), (1,1,-1)\}$ of $\mathbb{R}^3$ is given by $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{pmatrix}$. Find the matrix of $T$ relative to the ordered basis $\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$. *[(MATH4126.5)(Evaluate/HOCQ)]*

**6 + 6 = 12**

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| Cognition Level | LOCQ | IOCQ | HOCQ |
|---|---|---|---|
| Percentage distribution | 15.63 | 51.04 | 33.33 |