ALGEBRAIC STRUCTURES
(MATH 2201)

Time Allotted : 2½ hrs **Full Marks : 60**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and <u>any 4 (four)</u> from Group B to E, taking <u>one</u> from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A

1. Answer any twelve: **12 × 1 = 12**

Choose the correct alternative for the following

(i) Let $f(x) = y$, where $f(x) = \sqrt{x}$ and $x, y \in \mathbb{R}$. Which of the following is correct?
(a) $f(x)$ is an injective function but not a bijective function.
(b) $f(x)$ is a surjective function but not a bijective function.
(c) $f(x)$ is a bijective function.
(d) $f(x)$ is not a function.

(ii) In the group $\{1, -1, i, -i\}$ under multiplication the order of $-i$ is?
(a) 0 (b) 2 (c) 4 (d) -4.

(iii) Let $(X, \leq)$ be a poset, where X is the set of divisors of 60 and the relation defined on it, is the divisibility relation. Which of the following is true?
(a) $2 \leq 3$ (b) $2 \nleq 3$ (c) $3 \leq 2$ (d) $2 = 3$.

(iv) Which of the following permutation is cyclic?
(a) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$
(c) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$.

(v) If G is a cyclic group of order 8 with generator x, then
(a) x^4 is also a generator of G (b) x^5 is also a generator of G
(c) x^6 is also a generator of G (d) x^2 is also a generator of G.

(vi) Which of the following algebraic structure is not a group?
(a) $(\mathbb{Z}_7, +)$ (b) $(\mathbb{Z}_7, \cdot)$ (c) $(\mathbb{Z}_6, +)$ (d) $(\mathbb{Z}_6, \cdot)$.

(vii) The order of the element $[2]$ in the group $(\mathbb{Z}_9, +)$ is
(a) 0 (b) 3 (c) 9 (d) ∞

(viii) The number of homomorphisms from $\mathbb{Z}_4$ to $\mathbb{Z}_{12}$ is
(a) 4 (b) 3 (c) 12 (d) 48.

(ix) If G be a group of order 7, then G is necessarily a
(a) non-abelian group (b) cyclic group
(c) non-cyclic group (d) symmetric group.

(x) Which of the following is an example of Integral Domain?
(a) $\mathbb{Z}_4$ (b) $\mathbb{Z}_6$ (c) $\mathbb{Z}_7$ (d) $\mathbb{Z}_{10}$.

Fill in the blanks with the correct word

(xi) A relation which is reflexive, symmetric and transitive is known as _________.

(xii) The order of the group S_6 under composition of permutations is _________.

(xiii) The inverse of $[6] \in (\mathbb{Z}_7, .)$ is _________.

(xiv) Characteristic of Ring $\mathbb{Z}_4$ is _________.

(xv) A commutative ring R with unity and no zero divisors is called _________.

Group - B

2\. (a) Consider $A_1 = \{1, 2, 3, 4\}$, $A_2 = \{4, 5, 6\}$, and $A_3 = \{6, 7, 8\}$. Let r_1 be the relation on $A_1 \times A_2$ defined by $r_1 = \{(x, y) | y - x = 2\}$, and let r_2 be the relation on $A_2 \times A_3$ defined by $r_2 = \{(x, y) | y - x = 1\}$.
(i) Determine the adjacency matrices of r_1 and r_2.
(ii) Find the adjacency matrix of the composition $r_1 r_2$.
[(MATH2201.1,MATH2201.5)(Apply/IOCQ)]

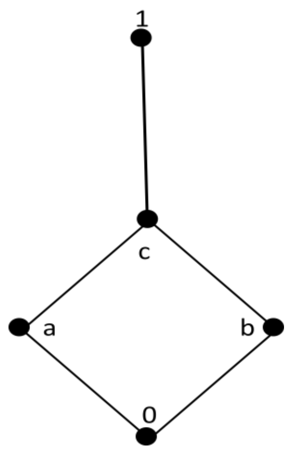
(b) Let ρ be a relation on the set $\mathbb{C}$ (set of complex numbers) and is defined by $(a_1 + i\, b_1)\ \rho\ (a_2 + i\, b_2)$ if and only if $a_1 \leq a_2$ and $b_1 \leq b_2$ for all $(a_1 + i\, b_1)$, $(a_2 + i\, b_2) \in \mathbb{C}$. Prove that $(\mathbb{C}, \rho)$ is a poset. *[(MATH2201.1,MATH2201.5)(Evaluate/HOCQ)]*

6 + 6 = 12

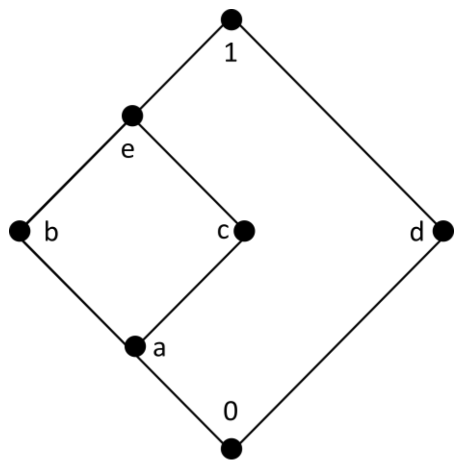
3\. (a) Consider the poset $A = \{3, 5, 9, 15, 24, 45\}$ with the divisibility relation defined on A.
(i) Draw its Hasse diagram.
(ii) Find its maximum, minimum, maximal and minimal elements.
[(MATH2201.1,MATH2201.5)(Create/HOCQ)]

(b) Are the following lattices distributive or not? Explain.

(i)



(ii)



[(MATH2201.1,MATH2201.5)(Analyse/IOCQ)]

6 + 6 = 12

Group - C

4. (a) Show that a group $(G,*)$ is an abelian group if and only if $(a * b)^{-1} = a^{-1} * b^{-1} \;\forall\, a, b \in G$.

[(MATH2201.2,MATH2201.3,MATH2201.4,MATH2201.6)(Analyze/IOCQ)]

(b) Show that $S = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : a^2 + b^2 = 1 \;\forall a, b \in \mathbb{R} \right\}$ forms a group with respect to matrix multiplication. Is this group abelian? Justify your answer.

[(MATH2201.2, MATH2201.3, MATH2201.4,MATH2201.6)(Evaluate/HOCQ)]

6 + 6 = 12

5. (a) Consider the following permutations in S_5: $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 6 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 2 & 1 & 6 \end{pmatrix}$ and $\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 1 & 5 \end{pmatrix}$.

Compute: (i) $\alpha\beta$ (ii) $\gamma\alpha$ and (iii) $\gamma\alpha\gamma^{-1}$.

[(MATH2201.2,MATH2201.3,MATH2201.4,MATH2201.6)(Remember/LOCQ)]

(b) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 4 & 7 & 5 & 2 & 3 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 6 & 7 & 3 & 5 & 2 \end{pmatrix}$ be elements of S_7.

(i) Write α as a product of disjoint cycles.
(ii) Is β an even permutation? Justify your answer.
(iii) Determine whether α^{-1} is an even or odd permutation.

[(MATH2201.2,MATH2201.3,MATH2201.4,MATH2201.6)(Analyze/IOCQ)]

6 + 6 = 12

Group - D

6. (a) Let $(G,*)$ be a group and $a \in G$, such that $O(a) = n$. Then show that $a^m = e$ (the identity in G) if and only if n divides m.

[(MATH2201.2,MATH2201.3,MATH2201.4,MATH2201.6)(Evaluate/HOCQ)]

(b) Prove Lagrange's theorem: The order of each subgroup of a finite group is a divisor of the order of the group.

[(MATH2201.2,MATH2201.3,MATH2201.4,MATH2201.6)(Remember/LOCQ)]

6 + 6 = 12

7. (a) Let G be a group and H, K be two subgroups of G. Prove that $H \cap K$ is a subgroup of G. Is $H \cup K$ is also a subgroup of G? Justify your answer.

[(MATH2201.2,MATH2201.3,MATH2201.4,MATH2201.6)(Understand/LOCQ)]

(b) Show that every subgroup of a cyclic group is a normal subgroup.

[(MATH2201.2,MATH2201.3,MATH2201.4,MATH2201.6)(Analyse/IOCQ)]

(c) Find all the generators of the cyclic group $\mathbb{Z}_7$ with respect to multiplication of residue classes modulo 7. *[(MATH2201.2,MATH2201.3,MATH2201.4,MATH2201.6)(Understand/LOCQ)]*

6 + 3 + 3 = 12

Group - E

8. (a) Let $(\mathbb{R}, +)$ be the additive group of real numbers and $(\mathbb{C}, +)$ be the additive group of complex numbers. Define $\phi: (\mathbb{C}, +) \to (\mathbb{R}, +)$ as $\phi(x + iy) = x$. Show that ϕ is a homomorphism and hence determine $Ker\ \phi$ and $Im\ \phi$.

[(MATH2201.2,MATH2201.3,MATH2201.4)(Evaluate/HOCQ)]

(b) Show that the ring of matrices $\left\{\begin{pmatrix} 3a & 0 \\ 0 & 3b \end{pmatrix} : a, b \in \mathbb{Z}\right\}$ contains divisors of zero and does not contain the unity. *[(MATH2201.2,MATH2201.3,MATH2201.4)(Analyse/IOCQ)]*

6 + 6 = 12

9. (a) If in a ring R every $x \in R$ satisfies $x^2 = x$, prove that $x + x = 0, \forall\, x \in R$.
[(MATH2201.2,MATH2201.3,MATH2201.4)(Analyse/IOCQ)]

(b) Prove that the ring $(\mathbb{Z}_n, +, .)$ is an integral domain if and only if n is prime.
[(MATH2201.2,MATH2201.3,MATH2201.4)(Evaluate/HOCQ)]

6 + 6 = 12

Cognition Level	LOCQ	IOCQ	HOCQ
Percentage distribution	21.88	40.62	37.5

Course Outcome (CO):

After the completion of the course students will be able to

MATH2201.1 Describe the basic foundation of computer related concepts like sets, Posets, lattice and Boolean Algebra.

MATH2201.2 Analyze sets with binary operations and identify their structures of algebraic nature such as groups, rings and fields.

MATH2201.3 Give examples of groups, rings, subgroups, cyclic groups, homomorphism and isomorphism, integral domains, skew-fields and fields.

MATH2201.4 Compare even permutations and odd permutations, abelian and non-abelian groups, normal and non-normal subgroups and units and zero divisors in rings.

MATH2201.5 Adapt algebraic thinking to design programming languages.

MATH2201.6 Identify the application of finite group theory in cryptography and coding theory.

**LOCQ: Lower Order Cognitive Question; IOCQ: Intermediate Order Cognitive Question; HOCQ: Higher Order Cognitive Question.*