LINEAR ALGEBRA
(MATH 4126)

Time Allotted : 2½ hrs **Full Marks : 60**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 4 (four) from Group B to E, taking one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A

1. Answer any twelve: **12 × 1 = 12**

Choose the correct alternative for the following

(i) Let $T: \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation and the dimension of $Ker\ T$ is 2. Then the dimension of $Im\ T$ is
(a) 1 (b) 2 (c) 3 (d) 4.

(ii) If S and T are two subspaces of a vector space V over a field, then which one of the following is subspace of V also?
(a) $S \cup T$ (b) $S \cap T$ (c) $S - T$ (d) $T - S$.

(iii) If $T: \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation and $Ker\ T = \{\theta\}$, then
(a) T is one-one mapping (b) T is one-to-one mapping
(c) T is onto mapping (d) T is onto mapping but not one-to-one.

(iv) The geometric multiplicity of the eigenvalue 0 of the matrix
$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ is
(a) 0 (b) 1 (c) 2 (d) 3

(v) The norm of $\alpha = (-1, 2, 3)$ in $\mathbb{R}^3$ with standard inner product is
(a) 14 (b) -14 (c) $\sqrt{14}$ (d) -6.

(vi) In a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ if $T\{(1,0),(0,1)\} = \{(2,7),(1,3)\}$, then the matrix representation of T with respect to standard basis of $\mathbb{R}^2$ is
(a) $\begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 \\ 7 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 3 \\ 7 & 1 \end{bmatrix}$ (d) Cannot be determined

(vii) If A be a singular matrix then an eigenvalue of A is
(a) 1 (b) 2 (c) -1 (d) 0

(viii) Let A be a non-zero 3×3 matrix such that $A^3 = A$. Then which of the following is true?
(a) A must be an identity matrix (b) A^2 must be an identity matrix
(c) A is invertible (d) A is diagonalizable.

(ix) If A is a 3×3 diagonalizable matrix, then the number of linearly independent eigenvectors of A is

(a) 1 (b) 2 (c) 3 (d) 4

(x) Let V be the vector space of all polynomials of degree less than or equal to 4. Then the dimension of V is

(a) 3 (b) 4 (c) 5 (d) infinity

Fill in the blanks with the correct word

(xi) If $u = (1,3,-4,2) \in \mathbb{R}^4$, then $\|u\|$ is _________.

(xii) If the vectors $(1,k,-3)$ and $(2,-5,4)$ are orthogonal then real value of k is _________.

(xiii) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ is a linear transformation and the dimension of $Im\,T$ be 2 then rank of T is _________.

(xiv) If a linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is defined by $T(x,y,z) = (x+z, y+z)$, then rank of T is _________.

(xv) The row vectors of an orthogonal matrix are _________________________.

Group - B

2. (a) Find the eigenvalues and the corresponding eigenvectors of the matrix given by $A = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$. Also, find the algebraic multiplicity and geometric multiplicity of each of the eigenvalues. *[(MATH4126.2,MATH4126.6)(Analyse/HOCQ)]*

(b) Show that the matrix $A = \begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix}$ is diagonalizable and also find the diagonal form. *[(MATH4126.2,MATH4126.6)(Apply/IOCQ)]*

6 + 6 = 12

3. (a) Let $A = \begin{pmatrix} 7 & 3 \\ 3 & -1 \end{pmatrix}$. Find an orthogonal matrix P such that $D = P^{-1}AP$. *[(MATH4126.1)(Apply/IOCQ)]*

(b) Find the Singular Value Decomposition of $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$. *[(MATH 4126.1)(Evaluate/HOCQ)]*

5 + 7=12

Group - C

4. (a) Determine whether the set of vectors $\{(2,-1,1), (2,0,3), (1,1,-2)\}$ forms a basis of the vector space $\mathbb{R}^3$ or not. *[(MATH 4126.2)(Understand/LOCQ)]*

(b) Find the real values of λ for which the following system of linear equations has non-trivial solutions:

$2\lambda x - 2y + 3z = 0,\ x + \lambda y + 2z = 0,\ 2x + \lambda z = 0$ *[(MATH4126.2)(Remember/LOCQ)]*

(c) Show that the subset $S = \{(x,y,z) \in \mathbb{R}^3 : 3x - y + z = 0\}$ is a subspace of $\mathbb{R}^3$. If yes, find a basis and dimension of S. *[(MATH4126.2)(Apply/IOCQ)]*

4 + 2 + 6 = 12

5. (a) Determine whether the set of vectors $\{(-1,3,-2),(2,4,1),(1,1,1)\}$ forms a basis of the vector space $\mathbb{R}^3$ or not. *[(MATH4126.2)(Understand/LOCQ)]*

(b) Consider the system of non-homogeneous linear equations $AX = B$ which is consistent. Now if the corresponding homogeneous system of linear equations $AX = O$ posses only the trivial solution, then discuss the nature of solutions of the given non-homogeneous system. *[(MATH4126.2)(Remember/LOCQ)]*

(c) Determine that the subset $S = \{(x, 2y, 3x) : x, y \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$ or not. If yes, find two basis of S. What is your conclusion about the dimension of S? *[(MATH4126.2)(Apply/IOCQ)]*

4 + 2 + 6 = 12

Group - D

6. (a) let $A = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. The inner product is defined as $\langle A, B \rangle = tr(B^T A)$. Find (i) $\langle A, B \rangle$, (ii) $\|A\|$ and (iii) $\|B\|$. *[(MATH4126.3, MATH4126.4)(Remember/LOCQ)]*

(b) Apply the Gram-Schmidt orthogonalization process to find an orthonormal basis for the subspace U of $\mathbb{R}^4$ spanned by $S = \{(1,1,1,1),(1,2,4,5),(1,-3,-4,-2)\}$. *[(MATH4126.3)(Apply/IOCQ)]*

6 + 6 = 12

7. (a) State and prove the Pythagorean theorem for norms. *[(MATH4126.3, MATH4126.4)(Remember/LOCQ)]*

(b) If u and v be two vectors in a real inner product space and $\|u\| = \|v\|$, then show that $\langle u + v, u - v \rangle = 0$. *[(MATH4126.3,MATH4126.4)(Evaluate/HOCQ)]*

(c) Let $\langle u, v \rangle$ be the standard inner product on $\mathbb{R}^2$. Let $\alpha = (1,2), \beta = (-1,1)$. If γ is a vector such that $\langle \alpha, \gamma \rangle = -1$ and $\langle \beta, \gamma \rangle = 3$, then find γ. *[(MATH4126.3, MATH4126.4)(Evaluate/HOCQ)]*

3 + 3 +6 = 12

Group - E

8. (a) A function $T: \mathbb{R}^3 \to \mathbb{R}^4$ is defined by $T(x, y, z) = (y + z,\ z + x,\ x + y,\ x + y + z)$ $\forall\ (x, y, z) \in \mathbb{R}^3$.
   (i) Show that T is a linear transformation.
   (ii) Find $Ker\ T$ and dimension of $Ker\ T$.
   (iii) Find $Im\ T$ and dimension of $Im\ T$.
   (iv) Verify Rank-Nullity Theorem for T. *[(MATH4126. 5)(Apply/IOCQ)]*

(b) A linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by
   $T(x, y, z) = (2x + y - z,\ y + 4z,\ x - y + 3z), \forall\ (x, y, z) \in \mathbb{R}^3$.
   Find the matrix of T relative to the standard ordered basis of $\mathbb{R}^3$. *[(MATH4126.5)(Evaluate/HOCQ)]*

8 + 4 = 12

9. (a) Find the linear transform $T: \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1,0,0) = (1,1)$, $T(0,1,0) = (2,3)$ and $T(0,0,1) = (3,2)$. Find $T(6,0,-1)$. Show that T is not one-to-one but onto. *[(MATH 4126.5)(Understand/LOCQ)]*

(b) If the matrix of a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ with respect to the ordered basis $B = \{(1,0), (0,1)\}$ be $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then find the matrix of T with respect to the ordered basis $B' = \{(1,1), (-1,1)\}$. *[(MATH4126.5)(Evaluate/HOCQ)]*

6 + 6 = 12

Cognition Level	LOCQ	IOCQ	HOCQ
Percentage distribution	28.13	38.54	33.33

Course Outcome (CO):

After the completion of the course students will be able to

MATH4126. 1. Explain concepts of diagonalization, orthogonal diagonalization and Singular Value Decomposition (SVD).

MATH4126. 2. Discuss basis, dimension and spanning sets.

MATH4126. 3. Design Gram-Schmidt Orthogonalization Process and QR decomposition using concepts of inner product spaces.

MATH4126. 4. Analyze Least squares solutions to find the closest line by understanding projections.

MATH4126. 5. Define linear transformations and change of basis.

MATH4126. 6. Illustrate applications of SVD such as, Image processing and EOF analysis, applications of Linear algebra in engineering with graphs and networks, Markov matrices, Fourier matrix, Fast Fourier Transform and linear programming.

**LOCQ: Lower Order Cognitive Question; IOCQ: Intermediate Order Cognitive Question; HOCQ: Higher Order Cognitive Question.*