B.TECH/IT/4TH SEM/MATH 2203 (BACKLOG)/2023

GRAPH THEORY AND ALGEBRAIC STRUCTURES
(MATH 2203)

Time Allotted : 3 hrs **Full Marks : 70**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 5 (five) from Group B to E, taking at least one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A
(Multiple Choice Type Questions)

1. Choose the correct alternative for the following: **10 × 1 = 10**

(i) The chromatic polynomial of K_3 is
(a) $\lambda(\lambda-1)(\lambda-2)$ (b) $\lambda(\lambda+1)(\lambda+2)$
(c) $\lambda(\lambda-1)$ (d) $\lambda(\lambda-1)^2$

(ii) The number of elements in the symmetric group S_5 is
(a) 5 (b) 20 (c) 120 (d) 55.

(iii) If H is a subgroup of a group G and a, b are two distinct elements of G, then indicate which of the following statements is true:
(a) $aH = Ha$ (b) $Ha \cap Hb = \phi$
(c) $Ha \cap Hb \neq \phi$ and $Ha \neq Hb$ (d) $aH = bH$.

(iv) The order of the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ is
(a) 2 (b) 3 (c) 1 (d) 4.

(v) The number of elements of order 6 in the cyclic group of order 42 is
(a) 7 (b) 2 (c) 3 (d) 42.

(vi) Index of a subgroup H of a group G is 5 and its order is 3. The order of the group G is
(a) 8 (b) 10 (c) 15 (d) 25.

(vii) The only generator(s) of the cyclic group $(\mathbb{Z}, +)$ is / are
(a) 1 (b) 0,1 (c) 1, −1 (d) none of these.

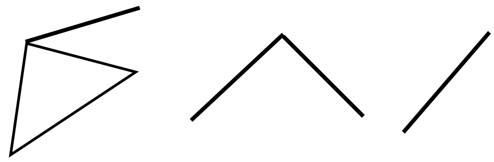
(viii) Consider the binary relation $R = \{(x,y),(x,z),(z,x),(z,y)\}$ on the set $\{x,y,z\}$. Which one of the following is TRUE?
(a) R is symmetric but not antisymmetric
(b) R is not symmetric but antisymmetric
(c) R is both symmetric and antisymmetric
(d) R is neither symmetric nor antisymmetric.

(ix) If x is an element of a group G and $O(x) = 5$, then
(a) $O(x^{10}) = 5$ (b) $O(x^{15}) = 5$
(c) $O(x^{23}) = 5$ (d) $O(x^{20}) = 5$

(x) The symmetric group S_3 is
(a) cyclic but not abelian (b) cyclic and abelian
(c) non cyclic and non-abelian (d) none of these.

Group - B

2. (a) Find the chromatic polynomial of the following disconnected graphG:



[(Evaluate/HOCQ)]

(b) Let G be a graph which has more than one edge. Prove that the sum of the coefficients in its chromatic polynomial is 0. *[(Remember/LOCQ)]*

6 + 6 = 12

3. (a) Prove that $\lambda^4 - 3\lambda^3 + 4\lambda^2$ cannot be a chromatic polynomial of a graph G. *[(Apply/IOCQ)]*

(b) How many edges a planar graph must have with 5 regions and 7 vertices? Draw one such graph. *[(Analyze/IOCQ)]*

6 + 6 = 12

Group - C

4. (a) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 6 \end{pmatrix}$
Compute each of the following: (i) α^{-1} (ii) $\alpha\beta$ *[(Analyze/IOCQ)]*

(b) Show that the set $G = \{(a + b\sqrt{2}) : a, b \in \mathbb{Q}\}$ is a group with respect to addition, where $\mathbb{Q}$ denotes the set of rationals. *[(Apply/IOCQ)]*

6 + 6 = 12

5. (a) Prove that the order of an element of a group is same as that of its inverse. *[(Remember/LOCQ)]*

(b) If G is a group such that $a^2 = e$ for all $a \in G$. Show that G is abelian. Is it true if $a^3 = e$, for all $a \in G$? *[(Understand/LOCQ)]*

6 + 6 = 12

Group - D

6. (a) Prove that every cyclic group is abelian. Is the converse true? Justify your answer. *[(Remember/LOCQ)]*

(b) If a be an element of order n in a group G and p be prime to n, then prove that a^p is also of order n. *[(Analyze/IOCQ)]*

6 + 6 = 12

7. (a) Prove that a group $(G,*)$ is commutative if $(a*b)^n = a^n * b^n$, for any three consecutive integers n and for all $a, b \in G$. *[(Understand/LOCQ)]*

(b) Let H be a subgroup of a group G and let $a, b \in G$. Prove that $aH = bH$ if and only if $a^{-1}b \in H$. *[(Apply/LOCQ)]*

6 + 6 = 12

Group - E

8. (a) Prove that the ring of matrices R=$\left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| a, b \in \mathbb{R}\right\}$ is a field. *[(Remember/LOCQ)]*

(b) Solve the equation $x^2 + 2x + 4 = 0$ in $\mathbb{Z}_6$. *[(Evaluate/HOCQ)]*

6 + 6 = 12

9. (a) Prove that every finite integral domain is a field. *[(Understand/LOCQ)]*

(b) Examine if the ring of matrices $\left\{\begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a, b \in \mathbb{R}\right\}$ is a field. *[(Apply/LOCQ)]*

6 + 6 = 12

Cognition Level	*LOCQ*	*IOCQ*	*HOCQ*
Percentage distribution	43.75	43.75	12.5

**LOCQ: Lower Order Cognitive Question; IOCQ: Intermediate Order Cognitive Question; HOCQ: Higher Order Cognitive Question*