ALGEBRAIC STRUCTURES
(MATH 2201)

Time Allotted : 3 hrs **Full Marks : 70**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 5 (five) from Group B to E, taking at least one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A
(Multiple Choice Type Questions)

1. Choose the correct alternative for the following: **10 × 1 = 10**

(i) Let $f(x) = x^2$, where $x \in \mathbb{R}$. Which of the following is correct?
(a) f^{-1} is an injective function but not a bijective function from $\mathbb{R}$ to $\mathbb{R}$.
(b) f^{-1} is a surjective function but not a bijective function from $\mathbb{R}$ to $\mathbb{R}$.
(c) f^{-1} is a bijective function from $\mathbb{R}$ to $\mathbb{R}$.
(d) f^{-1} is not a function from $\mathbb{R}$ to $\mathbb{R}$.

(ii) In the additive group $(\mathbb{Z}, +)$, $2^{-3} =$
(a) $\frac{1}{8}$ (b) -6 (c) 6 (d) $-\frac{1}{8}$.

(iii) In the group $(\mathbb{Z}_6, +)$, the order of $[4]$ is
(a) 4 (b) 3 (c) 2 (d) 1.

(iv) In the group $\mathbb{Z}_4$ under $+$ operation, $[2] + [3]$ is equal to
(a) [1] (b) [2] (c) [3] (d) [5].

(v) In the group $\{1, -1, i, -i\}$ under multiplication, the order of $-i$ is
(a) 0 (b) 2 (c) 4 (d) -4.

(vi) Let $(G, \star)$ be a group and $a \in G$ such that $o(a) = 20$, then the order of a^6 is
(a) 1 (b) 6 (c) 10 (d) 20.

(vii) Which of the following is a subgroup of the multiplicative group $\{1, -1, i, -i\}$?
(a) $\{1, i\}$ (b) $\{i, -i\}$ (c) $\{1, -1, i\}$ (d) $\{1, -1\}$.

(viii) Let G be a group of order 17. The number of subgroups of G is
(a) 1 (b) 2 (c) 3 (d) 4.

(ix) Let ϕ be a group homomorphism from $\mathbb{R}$ to $\mathbb{R}^+$ defined by $\phi(x) = e^{5x}\ \forall\, x \in \mathbb{R}$. Then $\phi(\log 3)$ is equal to
(a) 3^5 (b) 5^3 (c) e^{3x} (d) e^{5x}.

(x) The number of elements in the symmetric group S_5 is
(a) 24 (b) 120 (c) 5 (d) infinite.

Group - B

2. (a) Find the relation R on set A = {1,2,3,4}, whose matrix is given below:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Using matrix property check whether the relation is symmetric, anti-symmetric, or transitive? *[(MATH2201.1, MATH2201.5)(Understand/LOCQ)]*

(b) Consider the set $S = \mathbb{Z}$ where xRy if and only if $2|(x+y)$. Prove that R is an equivalence relation. *[(MATH2201.1, MATH2201.5)(Evaluate/HOCQ)]*

6 + 6 = 12

3. (a) For two elements $a, b \in \mathbb{R}$, $a \sim b$ if a^3 divides b^3. Prove that $(\mathbb{R}, \sim)$ is a poset. *[(MATH2201.1, MATH2201.5)(Analyze/IOCQ)]*

(b) Draw the Hasse diagram of the dual lattice of the lattice of divisors of 30 with respect to the divisibility relation. *[(MATH2201.1, MATH2201.5)(Create/HOCQ)]*

6 + 6 = 12

Group - C

4. (a) Show that in a group $(G, *)$, $(a * b)^{-1} = b^{-1} * a^{-1} \; \forall \, a, \; b \in G$. *[(MATH2201.2, MATH2201.3, MATH2201.4, MATH2201.6)(Analyze/IOCQ)]*

(b) Let $G = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a (\neq 0) \in \mathbb{R} \right\}$. Prove that G is an abelian group with respect to matrix multiplication. *[(MATH2201.2, MATH2201.3, MATH2201.4, MATH2201.6)(Evaluate/HOCQ)]*

6 + 6 = 12

5. (a) Prove that the identity element and the inverse of an element in a group is unique. *[(MATH2201.2, MATH2201.3, MATH2201.4, MATH2201.6)(Understand/LOCQ)]*

(b) Find the order of the following permutations. Justify your answers.
(i) (1 3 2)(5 4 6) in S_6 (ii) (1 2 5 3)(2 4 3 6 7) in S_7 (iii) (4 3 5)(2 3 5) in S_6. *[(MATH2201.2, MATH2201.3, MATH2201.4, MATH2201.6)(Analyse/IOCQ)]*

6 + 6 = 12

Group - D

6. (a) Let $(G, \star)$ be a group and $a \in G$ such that $o(a) = n$. Show that the $o(a) = o(a^{-1}) = n$. *[(MATH2201.2, MATH2201.3, MATH2201.4, MATH2201.6)(Apply/IOCQ)]*

(b) Prove that every cyclic group is an abelian group. Is the converse true? Justify your answer. *[(MATH2201.2, MATH2201.3, MATH2201.4, MATH2201.6)(Analyze/IOCQ)]*

6 + 6 = 12

7. (a) Let $\mathbb{Z}_{11}$ be the group of residue classes under multiplication modulo 11. Find the inverse of each element. Find the cyclic subgroup of $\mathbb{Z}_{11}$ generated by [2]. *[(MATH2201.2, MATH2201.3, MATH2201.4, MATH2201.6)(Understand/LOCQ)]*

(b) If $G = \langle a \rangle$ is a cyclic group of order 40, find all the distinct elements of the cyclic subgroup $\langle a^8 \rangle$. *[(MATH2201.2, MATH2201.3, MATH2201.4, MATH2201.6)(Apply/IOCQ)]*

(c) Show that a subgroup H of a group G is normal if and only if $xHx^{-1} = H \; \forall \; x \in G$.
[(MATH2201.2, MATH2201.3, MATH2201.4, MATH2201.6)(Apply/IOCQ)]

6 + 3 + 3 = 12

Group - E

8. (a) Let $(\mathbb{Z}, +)$ be the additive group of all integers and $(\mathbb{Q}^*, \times)$ be the multiplicative group of all non-zero rational numbers. Define $\phi: \mathbb{Z} \to \mathbb{Q}^*$ by $\phi(n) = 3^n, \forall \; n \in \mathbb{Z}$. Prove that ϕ is a homomorphism?
[(MATH2201.2, MATH2201.3, MATH2201.4)(Evaluate/HOCQ)]

(b) Let $(\mathbb{R}^*, \times)$ be the multiplicative group of all non-zero real numbers and $(\mathbb{C}^*, \times)$ be the multiplicative group of all non-zero complex numbers. Define a homomorphism $\phi: \mathbb{C}^* \to \mathbb{R}^*$ by $\phi(z) = |z|, \forall \; z \in \mathbb{C}^*$. Show that ϕ is neither monomorphism nor epimorphism.
[(MATH2201.2, MATH2201.3, MATH2201.4)(Evaluate/HOCQ)]

6 + 6 = 12

9. (a) Let $(\mathbb{R}, +)$ be the additive group of real numbers and let $(\mathbb{R}^*, \cdot)$ be the multiplicative group of non-zero real numbers. Prove that the homomorphism $\phi: \mathbb{R} \to \mathbb{R}^*$ given by $\phi(x) = e^x$ is a monomorphism and hence determine the $Ker\ \phi$ and $Im\ \phi$.
[(MATH2201.2, MATH2201.3, MATH2201.4)(Apply/IOCQ)]

(b) Let F be a field. Find all elements $a \in F$, such that $a = a^{-1}$.
[(MATH2201.2, MATH2201.3, MATH2201.4)(Evaluate/HOCQ)]

6 + 6 = 12

Cognition Level	*LOCQ*	*IOCQ*	*HOCQ*
Percentage distribution	*18.75*	*43.75*	*37.5*

Course Outcome (CO):

After the completion of the course students will be able to

MATH2201.1 Describe the basic foundation of computer related concepts like sets, POsets, lattice and Boolean Algebra.

MATH2201.2 Analyze sets with binary operations and identify their structures of algebraic nature such as groups, rings and fields.

MATH2201.3 Give examples of groups, rings, subgroups, cyclic groups, homomorphism and isomorphism, integral domains, skew-fields and fields.

MATH2201.4 Compare even permutations and odd permutations, abelian and non-abelian groups, normal and non-normal subgroups and units and zero divisors in rings.

MATH2201.5 Adapt algebraic thinking to design programming languages.

MATH2201.6 Identify the application of finite group theory in cryptography and coding theory.

**LOCQ: Lower Order Cognitive Question; IOCQ: Intermediate Order Cognitive Question; HOCQ: Higher Order Cognitive Question.*