LINEAR ALGEBRA
(MATH 4126)

Time Allotted : 3 hrs **Full Marks : 70**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 5 (five) from Group B to E, taking at least one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A
(Multiple Choice Type Questions)

1. Choose the correct alternative for the following: **10 × 1 = 10**

(i) If A and B are square matrices of order n and A is non-singular, then $A^{-1}B$ and BA^{-1} have
(a) same eigenvalues
(b) different eigenvalues
(c) distinct eigenvalues of same sign
(d) same eigenvalues of different sign.

(ii) For which value of θ the matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has real eigenvalues?
(a) 0 (b) π (c) 2π (d) all of the above.

(iii) A set of vectors in a vector space V over a field F is orthonormal if for any vector $\alpha \in V$,
(a) $\| \alpha \| = 0$ (b) $\| \alpha \| < 0$ (c) $\| \alpha \| > 0$ (d) $\| \alpha \| = 1$.

(iv) In a basis set B of a vector space V over a field F, the number of vectors in the basis set B is called the
(a) dimension of B
(b) dimension of F
(c) dimension of V
(d) all of the above.

(v) Which of the following set is an orthogonal set of vectors?
(a) $\{(0,3,4),(4,2,3),(0,0,1)\}$
(b) $\{(0,3,4),(1,0,0),(0,2,1)\}$
(c) $\{(0,3,4),(0,-4,3),(5,0,0)\}$
(d) All of the above.

(vi) Which among the following is not a basis of $\mathbb{R}^2$?
(a) $\{(1,0),(0,1)\}$
(b) $\{(1,1),(3,5)\}$
(c) $\{(1,1),(3,3)\}$
(d) $\{(1,0),(1,1)\}$.

(vii) Given the functions $S,T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $S(x,y) = (x,-y)$ and $T(x,y) = (xy, x+y)$, choose the appropriate statement.
(a) S is a linear mapping, but not T
(b) T is a linear mapping, but not S
(c) S and T both are linear mappings
(d) Neither S nor T is a linear mapping.

(viii) Which of the following pair of vectors is an orthogonal pair in $\mathbb{R}^2$ with respect to the inner product defined as $< x,y > = 3x_1y_1 + 2x_2y_2$ where $x = (x_1,x_2)$, $y = (y_1,y_2) \in \mathbb{R}^2$.
(a) $x = (1,1), y = (1,-1)$
(b) $x = (1,-1), y = (3,2)$
(c) $x = (1,-1), y = (2,3)$
(d) $x = (1,1), y = (-1,-1)$.

(ix) A linear mapping $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by $T(x_1, x_2) = (x_1 + x_2, x_1), \forall\, (x_1, x_2) \in \mathbb{R}^2$. The nullity of T is

(a) 3 (b) 2 (c) 1 (d) 0.

(x) If a linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is defined as $T(x, y, z) = (-x - z,\ y + z)$ then the rank of T is

(a) 1 (b) 2 (c) 3 (d) 4.

Group – B

2. (a) If possible, find a matrix P such that $P^{-1}AP$ is a diagonal matrix, where $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.
[(MATH4126.1, MATH4126.6)(Analyze/IOCQ)]

(b) Reduce the quadratic form $5x^2 + y^2 + 10z^2 - 4yz - 10zx$ to the normal form and show that it is positive definite. [(MATH4126.1, MATH4126.6)(Evaluate/HOCQ)]

6 + 6 = 12

3. (a) Find the singular value decomposition of the matrix $A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$.
[(MATH4126. 1, MATH4126. 6) (Evaluate/HOCQ)]

(b) If λ is an eigenvalue of a square matrix A, then show that λ^2 is an eigenvalue of A^2 .
[(MATH4126. 1, MATH4126. 6)(Analyze /IOCQ)]

8 + 4 = 12

Group – C

4. (a) Let $V = \{(a, b) |\ a, b\ \in \mathbb{R}\}$ be a set of vectors and F be a field of real numbers. Two operations are defined as follows: $(a_1, b_1) + (a_2, b_2) = (2b_1 + 2b_2, -a_1 - a_2)$ and $c(a, b) = (3cb, -ca)$. Show that under these two operations, V is not a vector space over F. [(MATH4126.2)(Analyze/IOCQ)]

(b) In a vector space V over a field F, if θ be the zero vector of V, then show that $\forall\ \alpha \in V$ and $\forall\ a\ \in F,\ a.\alpha = \theta$ implies that either $a = 0$ or $\alpha = \theta$.
[(MATH4126.2)(Understand/LOCQ)]

(c) What are the conditions on the scalar $a \in \mathbb{R}$, so that the vectors $(a, 1, 0)$, $(1, a, 1)$ and $(0, 1, a)$ in $\mathbb{R}^3$ are linearly independent? [(MATH4126.2)(Evaluate/HOCQ)]

4 + 2 + 6 = 12

5. (a) If $V(\mathbb{R})$ is a vector space of 2×3 matrices over $\mathbb{R}$, then determine whether the matrices $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & 3 \end{bmatrix}$ in $V(\mathbb{R})$ are linearly independent. [(MATH4126.2)(Analyze/IOCQ)]

(b) Determine if the set $S = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ forms a basis of the vector space of all 2×2 real matrices. Also, find the dimension of $span(S)$.
[(MATH4126.2)(Evaluate/HOCQ)]

6 + 6 = 12

Group – D

6. (a) Let $V(F)$ be a vector space of polynomials in t with inner product given by
$$\langle p, q\rangle = \int_0^1 p(t)q(t)dt, \text{ where } p(t),\ q(t) \in V.$$
Now for $p(t) = t + 2$ and $q(t) = t^2 - 2t - 3$, find (i) $\langle p, q\rangle$, (ii) $\|p\|$, (iii) $\|q\|$
[(MATH4126.3, MATH4126.4)(Remember/LOCQ)]

(b) Use Gram – Schmidt process to the vectors $(1, 0, 1), (1, 0, -1)$ and $(1, 3, 4)$ to obtain an orthogonal and corresponding orthonormal basis for $\mathbb{R}^3$ with the standard inner product. [(MATH4126.3, MATH4126.4)(Evaluate/HOCQ)]

6 + 6 = 12

7. (a) If u, v be vectors in a real inner product space and $\|u\| = \|v\|$, then show that $\langle u + v, u - v\rangle = 0$. [(MATH4126.3, MATH4126.4)(Understand/LOCQ)]

(b) Is the set $\left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}$ an orthonormal basis for $\mathbb{R}^2$? Justify your answer.
[(MATH4126.3, MATH4126.4)(Remember/LOCQ)]

(c) State and prove Schwarz's inequality for two vectors in an inner product space.
[(MATH4126.3, MATH4126.4)(Analyze/IOCQ)]

3 + 3 + 6 = 12

Group – E

8. (a) A function $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by
$T(x, y, z) = (x + y + z, 2x + y + 2z, x + 2y + z)\ \forall(x, y, z) \in \mathbb{R}^3$. Show that T is a linear transformation. Find $Ker\ T$ and the dimension of $Ker\ T$.
[(MATH4126.4, MATH4126.5)(Evaluate/HOCQ)]

(b) Verify that the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by
$T(x, y, z) = (x + y, y + z, z + x)\ \ \forall(x, y, z) \in \mathbb{R}^3$ is one-to-one and onto.
[(MATH4126. 4, MATH4126. 5) (Analyze/IOCQ)]

6 + 6 = 12

9. (a) Consider the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (3x + 4y, 2x - 5y$ and the basis of *ℝ2:S={1, 2, 2, 3}*. Find the matrix representing the linear transformation T relative to the basis S. [(MATH4126.5)(Evaluate/HOCQ)]

(b) A linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x, y, z) = (x - y, x + 2y, y + 3z$ *∀ x, y, z ∈ℝ*. Show that *T* is non-singluar and determine *T−1*.
[(MATH4126.5)(Evaluate/HOCQ)]

6 + 6 = 12

Cognition Level	LOCQ	IOCQ	HOCQ
Percentage distribution	14.58	33.34	52.08

Course Outcome (CO):

After the completion of the course students will be able to

CO1: Apply network theorems to solve electrical circuits having both dependent and independent sources.

CO2: Analyze magnetically coupled circuits.

CO3: Apply Laplace transform technique in solving transient problems of electrical circuits.

CO4: Apply the concept of graph theory to electrical circuits.

CO5: Obtain the equivalent representation of electrical circuits using two- port parameter representation.

CO6: Analyze and synthesize filters.

*LOCQ: Lower Order Cognitive Question; IOCQ: Intermediate Order Cognitive Question; HOCQ: Higher Order Cognitive Question.