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Discussion on Stability and Hopf-bifurcation of an Infected Prey under Refuge and Predator

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Abstract

The paper deals with the case of non-selective predation in a partially infected prey-predator system, where both the susceptible prey and predator follow the law of logistic growth and some preys avoid predation by hiding. The disease-free preys get infected in due course of time by a certain rate. However, the carrying capacity of the predator population is considered proportional to the sum-total of the susceptible and infected prey. The positivity and boundedness of the solutions of the system are studied and the existence of the equilibrium points and stability of the system are analyzed at these points. The effect of the infected prey-refuge on each population density is also discussed. It is observed that a Hopf-bifurcation may occur about the interior equilibrium, where the refuge parameter is considered as the bifurcation parameter. The analytical findings are illustrated through computer simulation using Maple that show the reliability of the model from the ecological point of view.

Keywords: Carrying capacity; Positivity; Boundedness; Susceptible; Stability; Bifurcation; Refuge

MSC 2010 No.: 92B05, 92B10

1. Introduction

The dynamical relationship between prey and predator has been of utmost importance in theoretical ecology and has been studied in detail by Zhao and Dai (2018), Kar and Mondol (2013) and Sarkar et al. (2017) to name a few. The predator functional response on prey population that describes the number of prey consumed per predator per unit of time is the most important element in prey-predator interaction. The most important functional responses is Lotka-Volterra functional response (Holling type-I functional response) that was pioneered by Holling (1965). After further studies, Holling (1965) along with Murray (1993) and Kot (2001) modified the concept and brought in the Holling type-II functional response that was studied in detail by Skalski and Gilliam (2001), Neverova et al. (2019) and others. Two species prey-predator models have been studied extensively in theoretical ecology for quite a long time. There are many research works on three species systems as well, e.g., two predators and one prey system studied by Freedman and Waltman (1977), Maity et al. (2008) and Kundu and Maitra (2018).

The effect of prey-refuges on the prey-predator interaction has always developed keen interest amongst researchers. The Australian fur seal feed on lobster and squid that often find refuge somewhere under water to escape predation. Several other examples are also available in the literature that provide reasonably convincing evidence that refuges can prohibit prey extinction, for example, Kar (2005), Sharma and Samanta (2015), Zhang et al. (2019), Abdulghafur and Naji (2018) and Kar et al. (2018) to name a few. These studies concluded that the refuge used by the prey has a stabilizing effect on the predator-prey interaction because once the prey escapes predation, the balance is maintained. Researchers had further improvised the interaction between multiple preys along with predators under the effect of prey refuge, i.e., some preys would find refuge somewhere in space to escape predation. Modelling of susecptible and infected prey together has also been an area of research, where both are subjected to predation.

In the present article we consider a system where the susceptible prey obeys the logistic growth rate which eventually gets infected by some diseases at a certain rate. Some common infections that marine creatures often have are Brooklynellosis, coral reef fish disease, etc. The predator population having a carrying capacity proportional to the size of the susceptible and infected prey together follows logistic law of growth. The proposed predator-prey model is an extension of the model suggested by Das et al. (2009). Chattopadhyay et al. (1999) has highlighted the aspect, where only the susceptible prey population is considered for predation but we have extended the model by considering that both susceptible and infected prey are equally vulnerable to predation because during predation it is not possible to discriminate between the susceptible and infected population and hence, the predator will consume both, which eventually makes our consideration more realistic.

The paper has been organized in eight sections with an appendix. The construction and model assumptions are discussed in Section 2. In Section 3, the positivity and boundedness of the system is discussed. Section 4 and Section 5 deal with the existence and the local along with the global stability analysis respectively. The existence of Hopf Bifurcation around the interior equilibrium has been shown in the next section. The important findings are numerically verified using Maple in Section 7. It is observed that our result supplements several other similar type of works. Finally, Section 8 contains the conclusive discussion and implications of our findings.

2. Formulation of the problem

Let us consider a prey-predator population that obeys the Holling-Tanner type dynamical system given as:

$$\begin{aligned} \frac{dx}{dt} &= r_1 x(1-\frac{x}{l}) - pxy - \alpha z(x-k_0), \\ \frac{dy}{dt} &= pxy - cy - qyz, \\ \frac{dz}{dt} &= r_2 z\{1-\frac{z}{v(x+y)}\} + m\alpha z(x-k_0), \end{aligned} \tag{1}$$

with initial conditions

$$x(0) \geq 0, y(0) \geq 0, z(0) \geq 0. \tag{2}$$

Here $x(t), y(t)$ and $z(t)$ are the densities of the susceptible prey, infected prey and predator population respectively at time t; r_1 is the intrinsic growth rate and l is the carrying capacity of the prey population; α is the predation parameter and k_0 is the number of preys that find refuge somewhere in space with $x > k_0$; m is the conversion factor in the case of consumption of susceptible prey population and p is the rate at which the prey population is infected known as the contact rate; c is the intrinsic death rate of the infected prey and q is the rate at which the predator population is consuming the infected prey.

Furthermore, the predator grows as per the logistic law with intrinsic growth rate r_2 and carrying capacity proportional to the size of the prey population. According to Das et al. (2009), $\frac{1}{v}$ is the amount of susceptible and infected prey required to support one predator at equilibrium when z equals $v(x+y)$. The dynamical behaviours of Holling-Tanner model is discussed in details in various articles like May (1974) and Braza (2018).

In the present article, $\alpha(x-k_0)$ is the amount of the susceptible prey consumed by one predator in a unit of time, a fraction m $(0 < m < 1)$ of the transformed energy goes into the reproduction of predators.

3. Basic Properties

The following theorem proves the positivity of system (1).

Theorem 3.1.

Every solution of system (1) with initial conditions (2) exists in the interval $[0, \infty)$ and $x(t) \geq 0,\ y(t) \geq 0,\ z(t) \geq 0$ for all $t \geq 0$.

Proof:

Since the right hand side of system (1) is completely continuous and locally Lipschitzian on C, the solution $(x(t), y(t), z(t))$ of (1) with initial conditions (2) exists and is unique on $[0, \xi)$, where $0 < \xi \leq +\infty$. From system (1) with initial conditions (2), we have:

$$x(t) = x(0)\exp\left[\int_0^t \left\{r_1 - \frac{r_1 x(\theta)}{l} - py(\theta) - \alpha z(\theta) + \frac{\alpha z(\theta) k_0}{x(\theta)}\right\} d\theta\right] \geq 0,$$
$$y(t) = y(0)\exp\left[\int_0^t \left\{px(\theta) - c - qz(\theta)\right\} d\theta\right] \geq 0,$$
$$z(t) = z(0)\exp\left[\int_0^t \left\{r_2 - \frac{r_2 z(\theta)}{v(x(\theta)+y(\theta))} + m\alpha(x(\theta) - k_0)\right\} d\theta\right] \geq 0.$$

Hence proved. ■

The following theorem proves the boudedness of system (1).

Theorem 3.2.

All the solutions of Equation (1) which initiate in R^3_+ are uniformly bounded if $\gamma < mc$, where γ is a positive constant to be suitably defined.

Proof:

Let us define a function

$$w = mx + my + z. \tag{3}$$

The derivative of Equation (3) with respect to time along with the solutions of (1) is

$$\frac{dw}{dt} = mr_1 x(1 - \frac{x}{l}) + r_2 z\{1 - \frac{z}{v(x+y)}\} - mcy - mqyz. \tag{4}$$

For each $\gamma > 0$, the following inequality holds:

$$\frac{dw}{dt} + \gamma w < \frac{ml}{4r_1}(r_1 + \gamma)^2 + \frac{v(x+y)(r_2+\gamma)^2}{4r_2} + m(\gamma - c)y. \tag{5}$$

If we consider $\gamma \leq c$, the above equation reduces to

$$\frac{dw}{dt} + \gamma w < \frac{ml}{4r_1}(r_1 + \gamma)^2 + \frac{v(x+y)(r_2+\gamma)^2}{4r_2}, \tag{6}$$

i.e.,

$$\frac{dw}{dt} + \left\{\gamma m - \frac{v(r_2+\gamma)^2}{4r_2}\right\} x + \left\{\gamma m - \frac{v(r_2+\gamma)^2}{4r_2}\right\} y + \gamma z < \frac{ml}{4r_1}(r_1+\gamma)^2. \tag{7}$$

Then, we can find an m' such that

$$\frac{dw}{dt} + \gamma m' x + \gamma m' y + \gamma z < \frac{ml}{4r_1}(r_1+\gamma)^2. \tag{8}$$

Since $0 < m < 1$ and $m' < m$, let

$$w' = m'x + m'y + z. \tag{9}$$

Therefore,

$$w' < w. \tag{10}$$

Now,

$$\frac{dw'}{dt} + \gamma w' < \frac{ml}{4r_1}(r_1 + \gamma)^2. \tag{11}$$

Consequently, it follows the right hand side of Equation (9) is bounded. Hence, we can find a μ such that

$$\frac{dw'}{dt} + \gamma w' < \mu. \tag{12}$$

Applying the theory of differential inequality, we obtain

$$0 < w'(x, y, z) < \frac{\mu}{\gamma}(1 - e^{-\gamma t}) + w'(x(0), y(0), z(0))e^{-\gamma t},$$

and for $t \to \infty$, we have

$$0 < w' < \frac{\mu}{\gamma}. \tag{13}$$

Since $w - w'$ is finite, let $w - w' \approx k$.

Hence, we have

$$k < w < \frac{\mu}{\gamma} + k. \tag{14}$$

Hence, all the solutions of Equation (1) that initiate in R^3_+ are confined in

$$B = \{(x, y, z) \in R^3_+ : k < w < \frac{\mu}{\gamma} + k, \text{ for any } k > 0\}.$$ ■

4. Equilibrium points: their existence and stability

In this section, we discuss the existence and stability behavior of the system (1) at the equilibrium points. The equilibrium points of (1) are:

(1) Trivial equilibrium: $E_0(0, 0, 0)$
(2) Axial equilibrium: $E_1(l, 0, 0)$

(3) Planar equilibrium:

(i) $E_2\left(\frac{c}{p}, \frac{r_1}{p}\left(1-\frac{c}{pl}\right), 0\right)$

(ii) $E_3(x_1, 0, z_1)$; $x_1 = k_0 + \frac{-B+\sqrt{B^2-4AC}}{2A}$ and $z_1 = \frac{r_1 x_1(l-x_1)}{l\alpha(x_1-k_0)}$;

$E_3(x_2, 0, z_2)$; $x_2 = k_0 + \frac{-B-\sqrt{B^2-4AC}}{2A}$ and $z_2 = \frac{r_1 x_2(l-x_2)}{l\alpha(x_2-k_0)}$,

where $A = ml\alpha^2 v$, $B = r_2 l\alpha v + r_1 r_2$ and $C = r_1 r_2(k_0 - l)$.

It is to be noted that $E_3(x_1, 0, z_1)$ exists if $x_1 > 0$ and either $l > x_1 > k_0$ or $l < x_1 < k_0$ and $E_3(x_2, 0, z_2)$ exists if $x_2 > 0$ and either $l > x_2 > k_0$ or $l < x_2 < k_0$.

(4) Interior equilibrium: $E_4(x^*, y^*, z^*)$

4.1. Existence of interior equilibrium

In this section, we will analyze the existence of non trivial interior equilibrium point of the model system (1). The following conditions hold at the interior equilibrium:

$$x > 0,\ y > 0,\ z > 0,$$

and

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0. \tag{15}$$

Solving Equation (1) at the equilibrium value, we get

$$z^* = \frac{px^* - c}{q},$$

and

$$y^* = \frac{r_1 x^* q(l - x^*) + \alpha(px^* - c)(x^* - k_0)}{pqx^*}.$$

Now, putting the values of z^* and y^* in the third equation of (1) at the interior equilibrium and simplifying, we get

$$Mx^{*3} + Nx^{*2} + Ox^* + P = 0, \tag{16}$$

where

$$M = vm\alpha r_1 q + vm\alpha^2 lp\ > 0,$$

$$N = vr_1r_2q + vr_2\alpha lp - vr_2pql - vm\alpha r_1ql - vm\alpha^2lpk_0$$
$$\text{-}vm\alpha^2lc - vm\alpha pql - vmk_0r_1q - vm\alpha lk_0p + vmk_0pql + p^2lr_2,$$

$$O = -vr_1r_2ql - vr_2\alpha lpk_0 - vr_2\alpha lc + vm\alpha^2lck_0$$
$$+vmk_0r_1ql + vm\alpha lk_0^2p + vm\alpha lck_0 - plr_2c,$$

$$P = vr_2\alpha lck_0 - vm\alpha lck_0^2 = v\alpha lkc_0(r_2 - \alpha k_0).$$

Therefore, if $P < 0$, i.e., the intrinsic growth rate of the predator is less than the product of the predation paramter and the number of preys that find refuge somewhere in space, there must exist at least one positive root of equation (16). A detailed calculation regarding this has been shown in Appendix 1. Summarizing the above observations, we arrive at the following result.

Theorem 4.1.

The necessary and sufficient condition for the existence of at least one non-trivial interior equilibrium point $E_4(x^*, y^*, z^*)$ of the system (1) is that P in Equation (16) must be negative.

4.2. Local Stability Analysis

4.2.1. *Trivial euilibrium* E_0

The variational matrix of the system (1) at $E_0(0,0,0)$ is given by

$$V(E_0) = \begin{bmatrix} r_1 & 0 & \alpha k_0 \\ 0 & -c & 0 \\ 0 & 0 & r_2 - m\alpha k_0 \end{bmatrix}.$$

Therefore, the eigenvalues of the characteristic equation of $V(E_0)$ are $\lambda_1 = r_1, \lambda_2 = -c$ and $\lambda_3 = r_2 - m\alpha k_0$. It is clear that λ_1 is positive making E_0 unstable. Hence, we arrive at the following theorem.

Theorem 4.2.

The trivial equilibrium of the system (1), although it exists, is unstable.

4.2.2. *Axial euilibrium* E_1

The variational matrix of system (1) at $E_1(l,0,0)$ is given by

$$V(E_1) = \begin{bmatrix} -r_1 & -pl & -\alpha l + \alpha k_0 \\ 0 & pl & 0 \\ 0 & 0 & r_2 + m\alpha(l - k_0) \end{bmatrix}.$$

Hence, the eigenvalues of the characteristic equation of $V(E_1)$ are $\lambda_1 = -r_1 < 0, \lambda_2 = pl > 0, \lambda_3 = r_2 + m\alpha(l - k_0)$. It is clear that λ_2 is positive making E_1 an unstable equilibrium. Therefore, we arrive at the following theorem.

Theorem 4.3.

The axial equilibrium of system (1) exists but is unstable.

4.2.3. *Planar equilibrium E_2 and E_3*

(i) E_2 exists only when $1 - \frac{c}{pl} > 0$, i.e., $l > \frac{c}{p}$.

We find that $\frac{dy}{dt}|_{z=0} < 0$ if $y(px - c) < 0$, i.e., $x_0 < \frac{c}{p}$. Since $x \leq x_0$ at any time t_0, where x_0 is the initial susceptible prey population, we have

$$x \leq x_0 \leq \frac{c}{p},$$

so that

$$px - c < 0.$$

Hence, $\frac{dy}{dt} < 0$, for all t, whenever $x_0 < \rho$, considering $\rho = \frac{c}{p}$.

Now, ρ is called the relative removal rate of the susceptible prey due to infection and hence, the infection in the susceptible prey population can not spread at all unless $x_0 > \rho$, known as the threshold phenomenon. Consequently, the carrying capacity of the susceptible prey population must exceed its relative removal rate due to infection.

Now, the variational matrix of the system (1) at $E_2\left(\frac{c}{p}, \frac{r_1}{p}(1 - \frac{c}{pl}), 0\right)$ is given by

$$V(E_2) = \begin{bmatrix} -\frac{r_1 c}{lp} & -c & -\frac{\alpha c}{p} + \alpha k_0 \\ r_1(1 - \frac{c}{pl}) & 0 & \frac{qr}{p}(1 - \frac{c}{pl}) \\ 0 & 0 & r_2 + m\alpha(\frac{c}{p} - k_0) \end{bmatrix}.$$

The eigenvalues of the characteristic equation of $V(E_2)$ are

$$\lambda_1 = r_2 + m\alpha\left(\frac{c}{p} - k_0\right),$$

and

$$\lambda_{2,3} = \frac{-\dfrac{r_1 c}{pl} \pm \sqrt{\dfrac{r_1^2 c^2}{p^2 l^2} - 4\left(cr_1 - \dfrac{c^2 r_1}{pl}\right)}}{2},$$

($\lambda_{2,3}$ indicate λ_2 and λ_3 for the positive and negative consideration in $\pm$, respectively).

Now, if $\frac{c}{p} > k_0$, i.e., the relative removal rate exceeds the number of of preys that somewhere find refuge in space, the equilibrium point becomes unstable and this is also the condition that the infection can not spread. Hence, we arrive at the following theorem.

Theorem 4.4.

If the infection in the prey population can not spread, the planar equilibrium point E_2 of Equation (1) becomes unstable.

(ii) $E_3(x_1, 0, z_1)$ or $E_3(x_2, 0, z_2)$ exist only when $x_1 > 0$ and $x_2 > 0$, i.e.,

$$k_0 + \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} > 0, \; z_1 > 0 \text{ and } z_2 > 0,$$

implying

$$\frac{r_1 x_1 (l - x_1)}{l\alpha(x_1 - k_0)} > 0,$$
$$\frac{r_1 x_2 (l - x_2)}{l\alpha(x_2 - k_0)} > 0.$$

Now, the variational matrix of the system (1) at $E_3(x_1, 0, z_1)$ is given by

$$V(E_3) = \begin{bmatrix} -r_1 - \dfrac{2r_1 x_1}{l} - \alpha z_1 & -px_1 & -\alpha x_1 + \alpha k_0 \\ 0 & px_1 - c - qz_1 & 0 \\ \dfrac{r_2 z_1^2}{v x_1^2} + m\alpha z_1 & \dfrac{r_2 z_1^2}{v x_1^2} & r_2 - \dfrac{2r_2 z_1}{v(x_1 + y_1)} + m\alpha(x_1 - k_0) \end{bmatrix}.$$

Therefore, the characteristic equation of $V(E_3)$ is given by

$$\lambda^2 + C\lambda + D = 0, \tag{17}$$

where

$$C = -2r_1 + 2r_1\left(\frac{x_1}{l} + \frac{z_1}{vx_1}\right) + \alpha z_1 - m\alpha(x_1 - k_0),$$

and

$$D = \left\{r_1 - \frac{2r_1x_1}{l} - \alpha z_1\right\}\left\{r_1 - \frac{2r_1z_1}{vx_1} + m\alpha(x_1 - k_0)\right\}.$$

By Routh-Hurwitz criterion, it follows that all the eigenvalues of the characteristic equation (17) have negative real parts if and only if

$$C > 0, D > 0. \tag{18}$$

Similar conclusion follows for $E_3(x_2, 0, z_2)$.

Hence, it follows that the system (1) shows a local asymptotic stability at E_3 when $x_1 > 0$, $x_2 > 0$, $z_1 > 0$, $z_2 > 0$ and condition (18) are simaltaneously satisfied.

So, we arrive at the following theorem.

Theorem 4.5.

The planar equilibrium E_3 of the system (1) exists and is locally aymptotically stable if $x_1 > 0$, $x_2 > 0$, $z_1 > 0$, $z_2 > 0$ and condition (18) are satisfied.

4.2.4. *Interior equilibrium E_4*

The variational matrix of (1) at $E_4(x^*, y^*, z^*)$ is given by

$$V(E_4) = \begin{bmatrix} -r_1 - \dfrac{2r_1x^*}{l} - py^* - \alpha z^* & -px^* & -\alpha x^* + \alpha k_0 \\ py^* & px^* - c - qz^* & -qy^* \\ \dfrac{r_2z^*}{v(x^*+y^*)^2} + m\alpha z^* & \dfrac{r_2z^*2}{v(x^*+y^*)^2} & r_2 - \dfrac{2r_2z^*}{v(x^*+y^*)} + m\alpha(x^* - k_0) \end{bmatrix}.$$

Therefore, the characteristic equation of $V(E_4)$ is given by,

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{19}$$

where

$$a_1 = r_1 - 2r_1x^* - py^* - \alpha z^* + r_2 - \frac{2r_2z^*}{v(x^*+y^*)} + m\alpha(x^* - k_0) + px^* - c - qz^*,$$

$$a_2 = -(r_1 - 2r_1x^* - py^* - \alpha z^*)\{r_2 - \frac{2r_2z^*}{v(x^*+y^*)} + m\alpha(x^* - k_0)$$
$$+px^* - c - qz^*\} - (px^* - c - qz^*)\{r_2 - \frac{2r_2z^*}{v(x^*+y^*)}$$
$$+m\alpha(x^* - k_0)\} - p^2x^*y^* + \alpha(k_0 - x^*)\{\frac{r_2z^*}{v(x^*+y^*)^2} + m\alpha z^*\},$$

$$a_3 = (r_1 - 2r_1x^* - py^* - \alpha z*)\{(px^* - c - qz^*) * (r_2 - \frac{2r_2z^*}{v(x^*+y^*)}$$
$$+m\alpha(x - k_0)) + \frac{qy^*r_2z^{*2}}{v(x^*+y^*)^2}\} + px^*\{py^*r_2 - py^*\frac{2r_2z^*}{v(x^*+y^*)}$$
$$+py^*m\alpha(x^* - k_0) + \frac{qy^*r_2z^*}{v(x^*+y^*)^2} + qy^*m\alpha z^*\} + (\alpha k_0 - \alpha x^*)\frac{py^*r_2z^{*2}}{v(x^*+y^*)^2}$$
$$-\alpha(k_0 - x^*)\{\frac{r_2z^*}{v(x^*+y^*)^2} + m\alpha z^*\}(px^* - c - qz^*).$$

By Routh-Hurwitz criterion, it follows that all eigenvalues of the characteristic equation (19) have negative real parts if and only if

$$a_1 > 0,\ a_3 > 0,\ a_1a_2 > a_3. \tag{20}$$

Therefore, the interior equilibrium E_4 of the model (1) is locally asymptotically stable when conditions (20) are satisfied. Hence, we have the following result.

Theorem 4.6.

The interior equilibrium E_4 of the system (1) is locally asymptotically stable if and only if conditions (20) are satisfied.

5. Global stability analysis

In this section, we shall study the global stability behaviour of the system at the interior equilibrium $E_4(x^*, y^*, z^*)$. Let us define

$$L = P[x - x^* - xln\frac{x}{x^*}] + Q[y - y^* - yln\frac{y}{y^*}] + R[z - z^* - zln\frac{z}{z^*}], \tag{21}$$

where P, Q, and R are positive constants to be chosen later.

It is to be noted that $L(x, y, z) \geq 0$ and $L(x^*, y^*, z^*) = 0$.

Furthermore, $\frac{dL}{dt}$ is negative definite and consequently L is a Lyapunov function studied by Murray (1993) and Kot (2001) with respect to all solutions in the positive octant. A detailed caculation in this regard has been shown in the Appendix 2.

Summarizing the above discussions we arrive at the following result.

Theorem 5.1.
If $\frac{r_1}{l} > \alpha k_0, z > z^*, y > y^*$ and $x^* + y^* > x + y$, then, E_4 is globally asymptotically stable.

6. Hopf-bifurcation at $E_4(x^*, y^*, z^*)$

The characteristic equation of the system (1) at E_4 is given by

$$\lambda^3 + a_1(m)\lambda^2 + a_2(m)\lambda + a_3(m) = 0, \tag{22}$$

where

$$a_1(m) = r_1 - 2r_1x^* - py^* - \alpha z^* + r_2 - \frac{2r_2z^*}{v(x^*+y^*)} + m\alpha(x^* - k_0) + px^* - c - qz^*,$$

$$\begin{aligned} a_2(m) = &-(r_1 - 2r_1x^* - py^* - \alpha z^*)\{r_2 - \frac{2r_2z^*}{v(x^*+y^*)} + m\alpha(x^* - k_0) \\ &+px^* - c - qz^*\} - (px^* - c - qz^*)\{r_2 - \frac{2r_2z^*}{v(x^*+y^*)} \\ &+m\alpha(x^* - k_0)\} - p^2x^*y^* + \alpha(k_0 - x^*)\{\frac{r_2z^*}{v(x^*+y^*)^2} + m\alpha z^*\}, \end{aligned}$$

$$\begin{aligned} a_3(m) = &(r_1 - 2r_1x^* - py^* - \alpha z*)\{(px^* - c - qz^*) * (r_2 - \frac{2r_2z^*}{v(x^*+y^*)} \\ &+m\alpha(x - k_0)) + \tfrac{qy^*r_2z^{*2}}{v(x^*+y^*)^2}\} + px^*\{py^*r_2 - py^*\frac{2r_2z^*}{v(x^*+y^*)} \\ &+py^*m\alpha(x^* - k_0) + \frac{qy^*r_2z^*}{v(x^*+y^*)^2} + qy^*m\alpha z^*\} + (\alpha k_0 - \alpha x^*)\frac{py^*r_2z^{*2}}{v(x^*+y^*)^2} \\ &-\alpha(k_0 - x^*)\{\frac{r_2z^*}{v(x^*+y^*)^2} + m\alpha z^*\}(px^* - c - qz^*). \end{aligned}$$

To check whether the system (1) is stable or not, let us consider m as the bifurcation parameter. For this purpose, using the following theorem stated by Murray (1993):

If $a_i(m),\ i = 1, 2, 3$ are smooth functions of m in an open interval about m_0 such that the characteristic equation (22) has

(1) a pair of complex eigenvalues $\lambda = \alpha(m) \pm i\beta(m)$(with $\alpha(m), \beta(m) \in R$) so that they become purely imaginary at $m = m_0$ and $\frac{d\alpha}{dm}\ |_{m=m_0} \neq 0$,

(2) the other eigenvalue is negative at $m = m_0$ then, a Hopf-bifurcation occurs around E_4 at $m = m_0$(i.e. a stability change of E_4 accompanied by the creation of a limit cycle at $m = m_0$).

Hence, we have the following result:

Theorem 6.1.

The system (1) possesses a Hopf-bifurcation around E_4 when m passes through m_0 provided $a_1(m_0) > 0,\ a_2(m_0) > 0$ and $a_1(m_0)a_2(m_0) = a_3(m_0)$.

Proof:

For $m = m_0$, the characteristic equation of the system (1) at E_4 becomes $(\lambda^2 + a_2)(\lambda + a_1) = 0$, providing roots $\lambda_1 = i\sqrt{a_2}$, $\lambda_2 = -i\sqrt{a_2}$, and $\lambda_3 = -a_1$. Hence, there exists a pair of purely imaginary eigenvalues and a strictly negative real eigenvalue. Also $a_i (i = 1, 2, 3)$ are smooth functions of m.

Taking m in a neighborhood of m_0, roots are of the form

$$\lambda_1(m) = b_1(m) + ib_2(m),$$

$$\lambda_2(m) = b_1(m) - ib_2(m),$$

$$\lambda_3 = -b_3(m),$$

where $b_i(m), i = 1, 2, 3$ are real.

Next, we verify the transversality condition

$$\frac{d}{dm}(Re(\lambda_i(m))) \mid_{m=m_0} \neq 0, i = 1, 2, 3.$$

Substituting $\lambda = b_1(m) + ib_2(m)$ into the characteristic equation (22), we get

$$(b_1 + ib_2)^3 + a_1(b_1 + ib_2)^2 + a_2(b_1 + ib_2) + a_3 = 0. \tag{23}$$

Taking the derivatives of both sides of (23) with respect to m, we get

$$\begin{aligned} 3(b_1 + ib_2)^2(\dot{b_1} + i\dot{b_2}) + 2a_1(b_1 + ib_2)(\dot{b_1} + i\dot{b_2}) \\ + \dot{a_1}(b_1 + ib_2)^2 + a_2(\dot{b_1} + i\dot{b_2}) + \dot{a_2}(b_1 + ib_2) + \dot{a_3} = 0. \end{aligned} \tag{24}$$

Comparing the real and imaginary parts from both sides of (24), we get,

$$D_1\dot{b_1} - D_2\dot{b_2} + D_3 = 0, \tag{25}$$

and

$$D_2\dot{b_1} + D_1\dot{b_2} + D_4 = 0, \tag{26}$$

where

$$\begin{gathered} D_1 = 3(b_1^2 - b_2^2) + 2a_1b_1 + a_2, \\ D_2 = 6b_1b_2 + 2a_1b_2, \end{gathered}$$

$$D_3 = \dot{a_1}(b_1^2 - b_2^2) + \dot{a_2}b_1 + \dot{a_3},$$
$$D_4 = 2\dot{a_1}b_1b_2 + \dot{a_2}b_2.$$

From (25) and (26) we get,

$$\dot{b_1} = -\frac{D_2D_4 + D_1D_3}{D_1^2 + D_2^2}. \tag{27}$$

Now,

$$D_3 = \dot{a_1}(b_1^2 - b_2^2) + \dot{a_2}b_1 + \dot{a_3} \neq \dot{a_1}(b_1^2 - b_2^2) + \dot{a_2}b_1 + \dot{a_1}a_2 + \dot{a_2}a_1.$$

At $m = m_0$:

Case I:

$$b_1 = 0, b_2 = \sqrt{a_2}.$$
$$D_1 = -2a_2, D_2 = -2a_1\sqrt{a_2}, D_3 \neq a_1\dot{a_2}, D_4 = \dot{a_2}\sqrt{a_2}.$$
$$\therefore D_2D_4 + D_1D_3 \neq 2a_1a_2\dot{a_2} - 2a_1a_2\dot{a_2} = 0.$$
So, $D_2D_4 + D_1D_3 \neq 0$ at $m = m_0$, when $b_1 = 0, b_2 = \sqrt{a_2}$.

Case II:

$$b_1 = 0, b_2 = -\sqrt{a_2}.$$
$$D_1 = -2a_2, D_2 = -2a_1\sqrt{a_2}, D_3 \neq a_1\dot{a_2}, D_4 = -\dot{a_2}\sqrt{a_2}.$$
$\therefore D_2D_4 + D_1D_3 \neq 2a_1a_2\dot{a_2} - 2a_1a_2\dot{a_2}$ and $2a_1a_2\dot{a_2} - 2a_1a_2\dot{a_2} = 0$.
So, $D_2D_4 + D_1D_3 \neq 0$ at $m = m_0$, when $b_1 = 0, b_2 = -\sqrt{a_2}$.
$$\therefore \frac{d}{dm}(Re(\lambda_i(m))) \mid_{m=m_0} = -\frac{D_2D_4 + D_1D_3}{D_1^2 + D_2^2} \mid_{m=m_0} \neq 0,$$

and

$$b_3(m_0) = -a_1(m_0) < 0.$$ ■

7. Numerical Results

Analytical studies remain incomplete without numerical validation of the theories proposed. Hence, in this section, we consider computer simulation of some solutions of system (1) where the values of the parameters have been set on the basis of data used in various other relevant research articles, e.g., Chattopadhyay et al. (1999), Neverova et al. (2019), Sharma and Samanta (2015) and Zhang et al. (2019). Apart from verifying our analytical findings, even from practical point of view, these numerical solutions are very important. We consider the values of the parameters in proper units in the examples mentioned below.

Example 7.1.

Considering the values in the data sets given in Table 1 and Table 2 below in appropriate units and plotting the susceptible and infected prey and predator population with respect to time, we find the curve and the phase space trajectories as in Figure 1, 2, 3, and 4 subsequently.

Table 1. Data Set 1 with parameter values

Parameter	**Values**
r_1	3
l	10
α	0.005
k_0	0.01
p	0.1
c	0.09
q	0.4
m	0.5
v	0.10
r_2	15

Table 2. Data Set 2 with parameter values

Parameter	**Values**
r_1	1.9
l	4
α	5
k_0	10
p	1
c	5
q	4
m	5
v	0.10
r_2	1.5

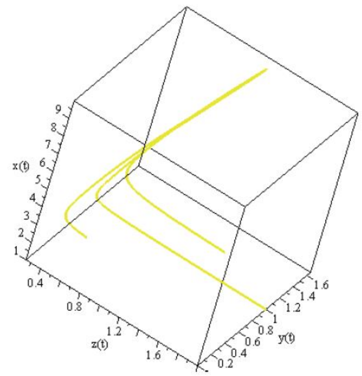

Figure 2. Phase space trajectories with parameter values of Table 1

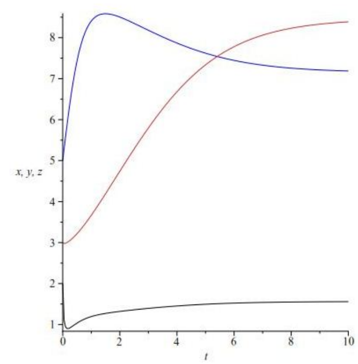

Figure 1. Time series plot of the susceptible, infected prey and predator population for parameter values as in Table 1

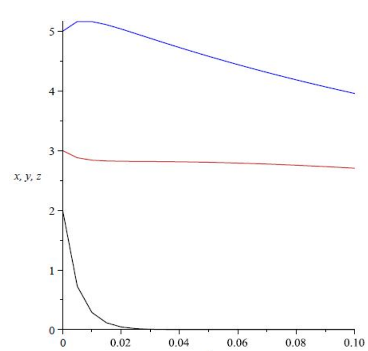

Figure 3. Time series plot of the susceptible, infected prey and predator population for parameter values as in Table 2

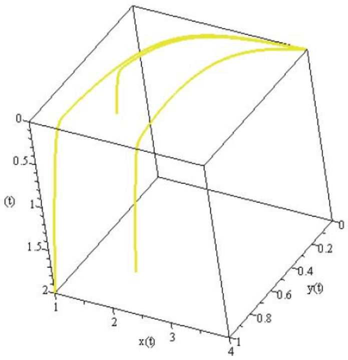

Figure 4. Phase space trajectories with parameter values of Table 2

8. Conclusion

A typical predator-prey model, where only the prey population is infected by an infectious disease, has been discussed thoroughly in this paper. The prey population has been divided into two categories, one that is susceptible and the other that is infected. It has been assumed that some susceptible preys manage refuge in space and hence avoid predation. The carrying capacity of the predator population is proportional to the cumulative total of susceptible and infected prey population. The dynamical behaviour of the system at various equilibrium points and their stability have also been discussed in detail. The proposed system has six equilibrium points, namely, one trivial equilibrium E_0, one axial equilibrium E_1, three planar equilibria E_2 and E_3 (with two sets of values) and an interior equilibrium point E_4. As observed, E_0 and E_1 always exist but are unstable. If the infection in the prey population can not spread then E_2 also becomes unstable. It has been found that the remaining planar equilibriums and the interior equilibrium are stable under certain conditions. The global stability of the interior equilibrium point has also been studied and the condition for which the interior equilibrium point is globally asymptotically stable has been obtained. However, the incorporation of prey refuge in the environment for which only a fraction of the susceptible prey remains accessible to the predator makes the paper more realistic. Increasing the amount of prey refuge can decrease both the infected prey density and the predator density. It has also been observed that the refuge parameter m is a very important factor to control the stability of the system and a stability switch and Hopf-bifurcation may occur at the interior equilibrium point taking m as the bifurcation parameter.

The other significant factor is the carrying capacity of the predator which is proportional to the infected and the susceptible prey population. Since analytical studies are incomplete without numerical verifications of the results, the findings are numerically verified using Maple. Hence, the prey-predator model described in the article shows very interesting dynamics when it is assumed that only the susceptible prey grows logistically but the infected prey does not. As the infected preys have disease in them, they are considered to be weak and hence incapable of reproduction. Therefore, the model can be further improvised considering the logistic growth in the infected population as well. One can also incorporate refuge in the infected population model. The harvesting of one or both the species may even be considered. However, the consumption of both susceptible and infected prey species by the predator is not an instantaneous process and hence there must be some time-lag known as gestation delay. Hence, there is ample scope of incorporating this delay in the model to make it even more realistic.

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APPENDIX

1. Conditions for Existence of Interior Equilibrium

It is obvious that M is always positive, However, the signs of $N,\ O,\ P$ are not obvious. Applying Descartes's Rule of Signs in Equation (4.2), we obtain:
1. if $N > 0, O > 0, P > 0$, then there is no change of sign, so there exists no positive root of Equation (16),
2. if $N > 0, O > 0, P < 0$, then there exists only one positive root of Equation (16),
3. if $N > 0, O < 0, P < 0$, then there exists only one positive root of Equation (16),
4. if $N < 0, O < 0, P < 0$, then there exists only one positive root of Equation (16),
5. if $N > 0, O < 0, P > 0$, then there exists two or no positive roots of Equation (16),
6. if $N < 0, O < 0, P > 0$, then there exists two or no positive roots of Equation (16),
7. if $N < 0, O > 0, P > 0$, then there exists two or no positive roots of Equation (16),
8. if $N < 0, O > 0, P < 0$, then there exists three or one positive roots of Equation(16).

2. Global Stability Analysis

Differentiating (21) along the solutions of (1) with respect to t we get,

$$\frac{dL}{dt} = P\frac{x - x^*}{x}\frac{dx}{dt} + Q\frac{y - y^*}{y}\frac{dy}{dt} + R\frac{z - z*}{z}\frac{dz}{dt}$$

$$= P[r_1(1 - \frac{x}{l}) - py - \alpha z + \frac{\alpha k_0}{x}](x - x^*) + Q[px - c - qz](y - y^*)$$
$$+ R[r_2\{1 - \frac{z}{v(x+y}\} + m\alpha(x - k_0)](z - z^*)$$

$$= P[\frac{r_1}{l}(x - x^*) - p(y - y^*) - \alpha(z - z^*) + \alpha k_0(\frac{1}{x} - \frac{1}{x^*})](x - x^*)$$
$$+ Q[p(x - x^*) - q(z - z^*)](y - y^*) + R[\frac{-r_2}{v}(\frac{1}{x+y} - \frac{1}{x^* + y^*})$$
$$+ m\alpha(x - x^*)](z - z^*)$$

$$= -P(\frac{r_1}{l} + \alpha k_0)\frac{(x - x^*)^2}{xx^*} + p(Q - P)(x - x^*)(y - y^*)$$
$$+ \alpha(Rm - P)(x - x^*)(z - z^*) + Qq(z - z^*)(y - y^*)$$
$$+ \frac{Rr_2}{v}\frac{x^* + y^* - x - y}{(x+y)(x^* + x)}(z - z^*).$$

Now, we choose $P = Q$ and $\frac{P}{R} = m$. Then, on simplification we get,

$$\frac{dL}{dt} = -P\left(\frac{r_1}{l} - \alpha k_0\right)\frac{(x - x^*)^2}{xx^*}$$
$$-(z - z^*)\frac{vQq(y - y^*)(x+y)(x^* + y^*) + Rr_2(x^* + y^* - x - y)}{v(x+y)(x^* + y^*)}.$$

Therefore, $\frac{dL}{dt} < 0$ if

$$\frac{r_1}{l} > \alpha k_0, z > z^*, y > y^* \text{ and } x^* + y^* > x + y.$$