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A multi-warehouse partial backlogging inventory model for deteriorating items under inflation when a delay in payment is permissible

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Abstract In this paper we develop a multi-item multi-warehouse inventory model for deteriorating items for m secondary warehouses (SWs) and one primary warehouse (PW) with displayed stock and price dependent demand under permissible delay in payment. Items are sold from PW which is located at the main market and due to large stock and insufficient space of existing PW, excess items are stored at m SWs of finite capacity. Due to different preserving facilities and storage environment, inventory holding cost is considered to be different in different warehouses. Here the demand of items is a deterministic function of corresponding selling price and the displayed inventory. Shortages are allowed and partially backlogged. The items of SWs are transported to the PW in continuous release pattern and associated transportation cost is proportional to the distance from PW to SWs. Here $M_i(< T_i$, cycle time) be the period of permissible delay in settling account for ith item, without the interest charges. But if the retailer settles the account after M_i, he will have to pay with interest per cycle for the inventory not sold after the due date M_i. A single objective inventory problem is solved numerically by developing Genetic algorithm and the maximum average profit and the corresponding optimum decision variables are evaluated. Finally the model is illustrated using a numerical example. A sensitivity analysis of the optimal solution with respect to the parameters of the system is carried out.

Keywords Inventory · Multi-warehouse · Delay in payment · Price and stock dependent demand · Deteriorating items · Genetic algorithm

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1 Introduction

In the field of inventory management, an important problem associated with the inventory maintenance is to decide where to stock the goods. This problem does not seem to have attracted much attention of researchers in this field. In this regard, the basic assumption in the traditional inventory models is that the management owns a store in primary warehouse with limited capacity. In the busy markets like super market, corporation market, municipality market, etc. the storage space of a primary warehouse is limited. We consider a multi-item multi-warehouse problem i.e. a system with one primary warehouse (PW), m secondary warehouses (viz., $SW_1, SW_2, \ldots, SW_m$) and N items. The warehouses $SW_1, SW_2, \ldots, SW_m$ contains $N_1, N_2, \ldots, N_m$ items respectively such that $N_1 + N_2 + \cdots, +N_m =$ N. Items are sold from PW which is located at the main market and shortages are allowed at this shop. Due to large stock and insufficient space of existing PW, excess items are stored at m secondary warehouses (viz., $SW_1, SW_2, \ldots, SW_m$) of finite capacity which are little away from PW. Here all the warehouses are of rental basis but rent of PW is greater than the rent of the secondary warehouses as they (SW's) are away from the main market place. Deterioration rates of an item are taken to be different in different warehouses. The stocks of SW's are transferred to PW under continuous release pattern and the associated transportation cost directly varies with the distance from PW to SWs but the holding cost of an item in SWs has reverse effect with distance. In realistic situations, some customers wait for backlogged items in stock-out period and hence partial loss in sale occurs. The backlogged demand is assumed to be a function of currently backlogged amount. Here, the problem is developed for a single time period.

Approximately thirty-three years ago, Hartely (1976) in U.S.A. first introduced two storage problems in his book "Operations Research-A Managerial Emphasis". In his analysis, he ignored the cost of transportation for transferring the goods from SW to PW. After Hartely (1976), Sarma (1983) in India first investigated two-storage inventory models in the year 1983. In his analysis, the following assumptions were made:

(i) Replenishment rate is infinite.
(ii) The items are transferred from SW to PW in a bulk release pattern taking constant transportation cost (This means that the transportation cost does not depend on the amount to be transported). Murdeshwar and Sathi (1985) extended this model by assuming a finite rate of replenishment. Dave (1989a) modified the EOQ models of Sarma (1983). Dave (1989a) considered finite as well as infinite rate of replenishment under some realistic assumptions and gave an algorithm for each model to get a complete solution. The above models were formulated for non-deteriorating items without allowing the shortages. Considering the shortages, Sarma (1987) formulated a model of deteriorating items with infinite replenishment rate. Next, Pakhala and Achary (1992a, b) developed the two warehouse models with finite rate of replenishment and shortages taking time as discrete and continuous variable respectively. In their models, the scheduling period was taken as constant and prescribed, and the transportation cost for transferring the stocks from SW to PW was not taken into account. All these models discussed earlier were developed for uniform demand only. Taking linearly time dependent demand, Goswami and Chaudhuri (1992) formulated two models for non-deteriorating items with or without shortages. Correcting and modifying the assumptions of Goswami and Chaudhuri (1992), Bhunia et al. (1994) analyzed the same and graphically presented a sensitivity analysis on the optimal average cost and the cycle length for the variations of the location and shape parameters of the demand. Trivedi and Shah (1994) discussed

a two-storage probabilistic problem for permissible delay in payment when the initial stock is a random variable. Bhunia and Maiti (1997, 1998), Lee and Ma (2000) studied a two-storage deterministic problem for deteriorating items considering linearly time-dependent demand and shortages. Benkherouf (1997) also investigated the same problem for deteriorating items taking time-dependent arbitrary demand rate function. In this model, he relaxed the assumptions of fixed cycle length and known quantity to be stocked in PW. Lastly, the model was illustrated with the help of an example where uniform linear and exponential type of demands ware considered instead of arbitrary time dependent demand rate. Kar et al. (2001) developed a two-storage inventory model with linearly time dependent demand over finite time horizon without lead-time and inflation. Recently Dey and Mondal (2008) developed a two storage inventory problem with dynamic demand and interval valued lead time over finite time horizon under inflation and time value of money.

In most of the stock-out inventory systems, either all the demand is back-ordered, in which all customers wait until their demands are satisfied; or the whole, demand is lost. However, in many realistic situations, during the stock-out period, the longer the waiting time is, the smaller the backlogging rate would be. For instance, for fashionable commodities, high-tech products with short life cycle, the willingness for a customer to wait for backlogging is diminishing with the length of the waiting time. The first paper in which customer impatience functions are proposed appears to be Abad's (1996). He developed an inventory model of dynamic pricing and lot-sizing by a retailer who sells perishable goods. Wee (1999) presents a deterministic inventory model with quantity discount, pricing and partial backlogging when the product in stock deteriorates with time. Chang and Dye (1999) developed an inventory model in which the proportion of customer who would like to accept backlogging is a reciprocal of a linear function of waiting time. Other recent articles relating to the research area were written by Abad (2001, 2003), Teng et al. (2003), Yang and Wee (2003), Teng and Yang (2004), Pal et al. (2006). As a consequence of high inflation, it is important to investigate how the time value of money influences various inventory policies. Following Buzacott (1975) and Misra (1979), several researchers (Hariga (1996), Datta and Pal (1991), Hariga and Ben-Daya (1996), Sarkar et al. (2000b), Horowitz (2000), Roy et al. (2007) etc) have extended their approaches and developed different inventory models by considering the time value of money, different inflation rates for internal and external costs, finite replenishment, shortages etc.

The inventory model under permissible delay in payments is among the extensions found in the literature. Goyal (1985) develops an inventory model under the condition of permissible delay in payments. Later, Aggarwal and Jaggi (1995) extend the Goyal (1985) model to consider an inventory model of deteriorating items with permissible delay in payments. Next, Jamal et al. (1997) further generalized the model to allow for shortages. Other authors also considered similar issues relating to delay in payments, e.g., Chu et al. (1998), Chung (1998), Sarkar et al. (2000a, b), Liao et al. (2000), Chang and Dye (2001), Teng (2002), Chung and Huang (2003), Salameh et al. (2003), Ouyand et al. (2005) etc. Recently, Huang (2007) established an EOQ model in which supplier offers a permissible delay in payment when the order quantity is smaller than the predetermined quantity. Though a considerable number of research work have been done in this area none has developed multi-warehouse inventory system for deteriorating item under inflation when delay in payment is permissible.

This paper is concerned with multi-item multi-warehouse problem for different deteriorating products with allowable shortage and permissible delay in payment. In this paper an attempt is made to formulate the mathematical model for deteriorating items with price and stock-dependent demand. A deterministic model is developed to optimize the net profit of a retailer. Genetic algorithm (GA) is used to maximize the profit function for optimal order cycle and order receipt period. A numerical example is given for illustration of the theoretical results, and sensitivity analysis for the profit function with respect to some parameters are carried out.

2 Genetic algorithm (GA)

The discovery of GA by Holland (1975) and further described by Goldberg (1998). GA is a randomized global search technique that solves problems imitating processes observed from natural evolution. GA continually exploits new and better solutions without any pre-assumptions such as continuity and unimodality. GA has been successfully adopted in many complex optimization problems and shows its merits over traditional optimization methods, especially when the system under study has multiple local optimal solutions. A GA normally starts with a set of potential solutions (called initial population) of the decision making problem under consideration. Individual solutions are called chromosomes. Crossover and mutation operations happen among the potential solutions to get a new set of solutions and it continues until terminating conditions are encountered. Michalewicz (1992) proposed a GA named Contractive Mapping Genetic Algorithm (CMGA) and proved the asymptotic convergence of the algorithm by Banach's fixed-point theorem. In CMGA, movement from old population to new takes place only when average fitness of new population is better than the old one. The algorithm is presented below. In the algorithm, p_c, p_m are probability of crossover and probability of mutation respectively, T is the generation counter and $P(T)$ is the population of potential solutions for generation T. M is iteration counter in each generation to improve $P(T)$ and M_0 is the upper limit of M. Initialize $(P(1))$ function generates the initial population $P(1)$ (initial guess of solution set). Objective function value due to each solution is taken as fitness of the solution. Evaluate $(P(T))$ function evaluates fitness of each member of $P(T)$.

GA algorithm

1. Set generation counter $T = 1$, iteration counter in each generation $M = 0$.
2. Initialize probability of crossover p_c, probability of mutation p_m, upper limit of iteration counter M_0, population size N.
3. Initialize $(P(T))$.
4. Evaluate $(P(T))$.
5. While $(M < M_0)$.
6. Select N solutions from P(T) for mating pool using roulette-wheel selection process Michalewicz (1992). Let this set be $P^{'}(T)$.
7. Select solutions from $P^{'}(T)$, for crossover depending on p_c.
8. Make crossover on selected solutions.
9. Select solutions from $P^{'}(T)$, for mutation depending on p_m.
10. Make mutation on selected solutions for mutation to get population $P_1(T)$.
11. Evaluate $(P_1(T))$.
12. Set $M = M + 1$.
13. If average fitness of $P_1(T) >$ average fitness of $P(T)$ then
14. Set $P(T + 1) = P_1(T)$.
15. Set $T = T + 1$.
16. Set $M = 0$.

17. End if
18. End while
19. Output: Best solution of $P(T)$.
20. End algorithm.

Procedures for different GA components

(a) Chromosome representation: The concept of chromosome is normally used in the GA to stand for a feasible solution to the problem. A chromosome has the form of a string of genes that can take on some value from a specified search space. The specific chromosome representation varies, based on the particular problem properties and requirements. Normally, there are two types of chromosome representation – (i) the binary vector representation based on bits, and (ii) the real number representation. In this research work, the real number representation scheme is used.
Here, a 'K dimensional real vector' $X = (x_1, x_2, \ldots, x_K)$ is used to represent a solution, where $x_1, x_2, \ldots, x_K$ represent different decision variables of the problem.
(b) Initialization: A set of solutions (chromosomes) is called a population. N such solutions $X_1, X_2, X_3, \ldots, X_N$ are randomly generated from search space by random number generator such that each X_i satisfies the constraints of the problem. This solution set is taken as initial population and is the starting point for a GA to evolve desired solutions. At this step, probability of crossover p_c and probability of mutation p_m are also initialized. These two parameters are used to select chromosomes from mating pool for genetic operations-crossover and mutation respectively.
(c) Fitness value: All the chromosomes in the population are evaluated using a fitness function. This fitness value is a measure of whether the chromosome is suited for the environment under consideration. Chromosomes with higher fitness will receive larger probabilities of inheritance in subsequent generations, while chromosomes with low fitness will more likely be eliminated. The selection of a good and accurate fitness function is thus a key to the success of solving any problem quickly. In this paper, value of an objective function due to the solution X, is taken as fitness of X. Let it be f(X).
(d) Selection process to create mating pool: Selection in the GA is a scheme used to select some solutions from the population for mating pool. From this mating pool, pairs of individuals in the current generation are selected as parents to reproduce offspring. There are several selection schemes, such as roulette wheel selection, local selection, truncation selection, tournament selection, etc. Here, roulette wheel selection process is used in different cases. This process consists of the following steps-

(i) Find total fitness of the population $F = \sum_{i=1}^{N} f(X_i)$
(ii) Calculate the probability of selection pr_i of each solution X_i by the formula $pr_i = f(X_i)/F$.
(iii) Calculate the cumulative probability qr_i for each solution X_i by the formula $qr_i = \sum_{j=0}^{i} pr_j$
(iv) Generate a random number 'r' from the range [0, 1].
(v) If $r < qr_1$ then select X_1 otherwise select $X_i (2 \le i \le N)$ where $qr_{i-1} \le r < qr_i$.
(vi) Repeat step (iv) and (v) N times to select N solutions from current population. Clearly one solution may be selected more than once.
(vii) Let us denote this selected solution set by $P^1(T)$.

(e) Crossover: Crossover is a key operator in the GA and is used to exchange the main characteristics of parent individuals and pass them on to children. It consists of two steps:

(i) Selection for crossover: For each solution of $P^1(T)$ generates a random number r from the range [0, 1]. If r $< p_c$, the solution is taken for crossover, where p_c is the probability of crossover.

(ii) Crossover process: Crossover taken place on the selected solutions. For each pair of coupled solutions Y_1, Y_2, a random number c is generated from the range [0, 1] and Y_1, Y_2 are replaced by their offspring's Y_{11} and Y_{21} respectively where $Y_{11} = cY_1 + (1-c)$ Y_2, $Y_{21} = cY_2 + (1-c)Y_1$, provided Y_{11}, Y_{21} satisfy the constraints of the problem.

(f) Mutation: The mutation operation is needed after the crossover operation to maintain population diversity and recover possible loss of some good characteristics. It also consists of two steps:

(i) Selection for mutation: For each solution of $P^1(T)$ generates a random number r from the range [0, 1]. If r $< p_m$, the solution is taken for mutation, where p_m is the probability of mutation.

(ii) Mutation process: To mutate a solution $X = (x_1, x_2, ., x_K)$, select a random integer r in the range [1, K]. Then replace x_r by randomly generated value within the boundary of rth component of X.

Following selection, crossover and mutation, the new population is ready for its next iteration, i.e., $P^1(T)$ is taken as population of new generation. With these genetic operations a simple GA takes the following form. In the algorithm T is iteration counter, $P(T)$ is the population of potential solutions for iteration T, Evaluate $(P(T))$ evaluates fitness of each members of $P(T)$.

(g) Implementation: With the above function and values the algorithm is implemented using C-programming language.

Convergence of the GA: Michalewicz (1992) proposed the algorithm and proved the asymptotic convergence of the algorithm by Banach's fixed-point theorem.

3 Particle swarm optimization (PSO)

In the basic PSO model, a swarm of m particles moving about in an D-dimensional real valued search space, the i-th particle is a D-dimensional vector, denoted as $X_i = (x_{i1}, x_{i2}, x_{i3}, \ldots, x_{iD})$, i = 1,2,3,..., m. The i-th particle's "flying" velocity is also a D-dimensional vector, denoted as $V_i = (v_{i1}, v_{i2}, v_{i3}, \ldots, v_{iD})$. Denote the best position of the i-th particle as $P_{besti} = (p_{i1}, p_{i2}, p_{i3}, \ldots, p_{iD})$, and the best position of the colony $P_{gbest} = (p_{g1}, p_{g2}, p_{g3}, \ldots, p_{gD})$. Each particle of the population modified its position and velocity according to the following formulas:

$$V_i^{t+1} = w * V_i^t + c_1 * rand * (P_{besti} - X_i^t) + c_2 * Rand * (P_{gbest} - X_i^t)$$
$$\chi_{id}^{t+1} = \chi_{id}^t + v_{id}^{t+1}$$

where w is the inertia weight factor, rand and Rand are uniform random value in the range [0,1], t is the current generation number, V_i^t and X_i^t are the current velocity and position of the particle respectively, P_{besti} is the best solution this particle has reached, P_{gbest} is the current global best solution of all the particles, c_1 and c_2 are learning factors, determining the influence of P_{besti} and P_{gbest}. The first part of the first formula is the inertia velocity of particle, which reflects the memory behavior of particle; the second part (the distance between the current position and the best position of the i-th particle) is "cognition" part, which represents the private thinking of the particle itself; the third part (the distance between the current position of the i-th particle and the best position of the colony) is the "social" part, showing the particle's behavior stem from the experience of other particles in the population. The particles find the optimal solution by cooperation and competition.

4 Assumptions and notations : [For *i*th items(i = 1,2,…,N) and *j*th secondary warehouse(j = 1,2,…,m)]

The following assumptions and notations are employed throughout this paper so as to develop the inventory model.

4.1 Assumptions

(i) Demand is price and stock dependent.
(ii) Rate of replenishment is infinite and the replenishment size is finite.
(iii) Shortages are allowed in primary warehouse(PW) and backlogged partially. The fraction of shortages back ordered is a differentiable and decreasing function of time 't', denoted by $\delta(t)$ where t is the waiting time up to the next replenishment, with $0 \leq \delta(t) \leq 1$, $\delta(0) = 1$ and $\delta(t) \to 0$ when $t \to \infty$.
(iv) The opportunity cost due to lost sale is the sum of the revenue loss and the cost of loss of goodwill.
(v) Deterioration rate is constant.
(vi) There is no quantity discount.
(vii) The inventory-planning horizon is infinite and consider inflation and time value of money for one cycle.
(viii) The inventory system involves N items.
(ix) The units will be sold from primary warehouse(PW) and the space in PW will be immediately filled up by shifting continuously from the secondary warehouses(SW_j's).
(x) Time tag between selling from PW and filling up its space by new units from SW_j's is negligible.
(xi) Transportation cost is taken for transporting units from SW_j to PW.

4.2 Notations (for *i*th item, i = 1, 2,…,N)

(i) C_{3i}= Replenishment cost per cycle.
(ii) C_{2i} = Shortage cost per unit per unit time.
(iii) C_{oi} = Opportunity cost due to lost sale per unit per unit time.
(iv) p_i = The selling price per unit item.
(v) C_i = The unit purchasing cost, with $C_i < p_i$.
(vi) C_{1i}^{PW} = The inventory carrying cost per unit per unit time in PW.
(vii) d_j = Distance of SW_j from PW.
(viii) $C_{1i}^{SW_j}=\frac{C_{1i}^{PW}}{d_j^{\mu}}$ = The inventory carrying cost per unit per unit time in SW_j, where $\mu \geq 0$.
(ix) S_i = Total stock of the system at t = 0.
(x) W_i = inventory at PW.
(xi) M_i = The period of permissible delay in settling the account i.e. the trade credit period.
(xii) I_p = The interest charged per dollar in stocks per unit time by the supplier when the retailer pays after M_i.

(xiii) I_e = The interest earned per dollar per unit time.
(xiv) $\delta(t)$ = The backlogging rate which is a decreasing function of the waiting time t. We assume that $\delta(t) = e^{-\lambda t}$, $\lambda \geq 0$ and t is the waiting time.
(xv) R_i = Partial backlogging amount at time T_i.
(xvi) t_{ki} = The time when inventory level is zero, k = 1 for SW_j and k = 2 for PW.
(xvii) T_i = The total time period for the cycle.
(xviii) θ_{ki} = Constant deterioration rate, $0 < \theta_{ki} < 1$, k = 1 for SW_j and k = 2 for PW.
(xix) $q_{ki}(t)$ = On hand inventory at any time t (≥ 0), k = 1 for SW_j and k = 2 for PW.
(xx) D_i = Demand rate function varying with $q_{ki}(t)$ and p_i.
(xxi) f = Inflation rate.
(xxii) r = Discount rate representing the time value of money.
(xxiii) R = r-f, present value of the nominal inflation rate.
(xxiv) i_e = Nominal interest earned at time t = 0 (dollars per dollar per unit time).
(xxv) $l_e = i_e - r$.
(xxvi) $I_e(t)$ = Rate of interest earned at time t, dollars per dollar per unit time.
(xxvii) i_p = Nominal interest paid per dollar per unit time at time t = 0.
(xxviii) $l_p = i_p - r$.
(xxix) $I_p(t)$ = Interest rate paid at time t, dollars per dollar per unit time.

5 Mathematical formulation

The behavior of inventory level in the multi-warehouse inventory system with stock and price dependent demand for deteriorating items is depicted in Fig. 1. The system involves one primary warehouse (PW), m secondary warehouses (viz. $SW_1, SW_2, .., SW_m$) and N items. The inventory system starts with a replenishment 'S_i' (for ith item) at time t = 0 out of which W_i is stored at PW and the rest $S_i - W_i$ is stored at SW_j (j = 1, 2,..., m).

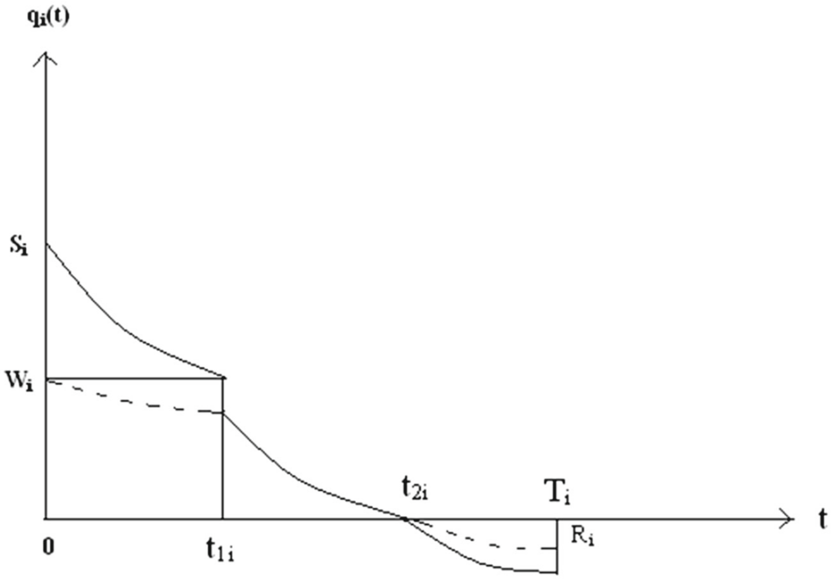

Fig. 1 Pictorial representation of the inventory behavior in a multi-warehouse system for deteriorating items

The goods of PW are consumed only after consuming the goods kept in SW_j's. During the interval $(0, t_{1i})$, the inventory $S_i - W_i$ in SW_j gradually decreases due to demand and deterioration and it vanishes at $t = t_{1i}$. In PW the inventory W_i decreases during $(0, t_{1i})$ due to deterioration only, but during (t_{1i}, t_{2i}) inventory depleted due to both demand and deterioration. At time t_{2i} all warehouses (PW and SW_j) are empty and thereafter shortages occur. The partially backlogged quantity is supplied to customers at the beginning of the next cycle. The demand rate is dependent on price only down to a fixed level, W_i, beyond which it is dependent on price and the on hand inventory and at the shortage period it is again dependent on price.

i.e.

$$D_i(p_i, q_{ki}) = \begin{cases} f(p_i, W_i), & \text{when } q_{ki} \geq W_i \\ f(p_i, q_{ki}), & \text{when } 0 \leq q_{ki} \leq W_i \\ f(p_i, 0), & \text{when } q_{ki} < 0 \end{cases}$$

k=1 for SW_j and k=2 for PW.

The objective of the model is to determine the timings t_{1i} and T_i so that the total average profit of the inventory system is maximized.

The total profit consist of :

(i) sales revenue,
(ii) cost of placing orders,
(iii) cost of purchasing.,
(iv) cost of carrying inventory,
(v) cost of backlogging,
(vi) opportunity cost due to lost sale,
(vii) cost of interest payable for items unsold after the permissible delay M_i,
(viii) interest earned from sales revenue during the permissible period,
(ix) transportation cost.

The differential equation describing the inventory level $q_{1i}(t)$, (i=1,2,…,N) for secondary warehouses (SW_j) of the system is

$$\frac{dq_{1i}(t)}{dt} = -D_i - \theta_{1i} q_{1i}(t) - \theta_{2i} W_i, \qquad 0 \leq t \leq t_{1i} \tag{1}$$

with boundary conditions

$$q_{1i}(t) = \begin{cases} sS_i - W_i & at\ t = 0 \\ 0 & at\ t = t_{1i}. \end{cases}$$

The differential equation describing the inventory level $q_{2i}(t)$, (i=1,2,…,N) for primary warehouse of the system is

$$\frac{dq_{2i}(t)}{dt} = \begin{cases} -(D_i + \theta_{2i} q_{2i}(t)), & t_{1i} \leq t \leq t_{2i} \\ -D_i.\delta(T_i - t), & t_{2i} \leq t \leq T_i \end{cases} \tag{2}$$

with boundary conditions

$$q_{2i}(t) = \begin{cases} W_i, & at\ t = t_{1i} \\ 0 & at\ t = t_{2i} \end{cases}$$

Here we assume

$$f(p_i, q_{ki}(t)) = \frac{\alpha_i + \gamma q_{ki}(t)}{p_i^{\beta}}, \quad where\ \alpha_i,\ \beta,\ \gamma > 0$$

Solving differential Eq. (1) we get

$$q_{1i}(t) = \frac{\alpha_i + (\gamma + \theta_{2i} p_i^{\beta}) W_i}{\theta_{1i} p_i^{\beta}} \left\{ e^{\theta_{1i}(t_{1i} - t)} - 1 \right\}, \quad where \quad 0 \le t \le t_{1i} \tag{3}$$

and solving differential Eq. (2) we get

$$q_{2i}(t) = \begin{cases} \frac{K_{2i}}{K_{1i}} \left(e^{K_{1i}(t_{2i} - t)} - 1 \right), & t_{1i} \le t \le t_{2i} \\ \frac{K_{2i}}{\lambda} \left(e^{-\lambda(T_i - t_{2i})} - e^{-\lambda(T_i - t)} \right), & t_{2i} \le t \le T_i \end{cases} \tag{4}$$

where $K_{1i} = \frac{\gamma + \theta_{2i} p_i^{\beta}}{p_i^{\beta}}$ and $K_{2i} = \frac{\alpha_i}{p_i^{\beta}}$.

Now with the boundary condition, $q_{1i}(0) = S_i - W_i$ and from (3) we get

$$S_i = W_i + K_{3i} \left(e^{\theta_{1i} t_{1i}} - 1 \right),$$

where $K_{3i} = \frac{\alpha_i + (\gamma + \theta_{2i} p_i^{\beta}) W_i}{\theta_{1i} p_i^{\beta}}$.

Now at $t = T_i$, $q_{2i}(T_i) = -R_i$ and from (4) we get

$$R_i = \frac{K_{2i}}{\lambda} \left\{ 1 - e^{-\lambda(T_i - t_{2i})} \right\}.$$

Integrating (1) between the limits $q_{1i}(t) = S_i - W_i$ to 0 when t=0 to $t = t_{1i}$ we get

$$\begin{aligned} t_{1i} &= \int_0^{S_i - W_i} \frac{dq_{1i}(t)}{\theta_{1i} q_{1i}(t) + f(p_i, W_i) + \theta_{2i} W_i} \\ &= \frac{1}{\theta_{1i}} \ln \left| \frac{\alpha_i + \theta_{1i} S_i p_i^{\beta} + \{\gamma + (\theta_{2i} - \theta_{1i}) p_i^{\beta}\} W_i}{\alpha_i + \{\gamma + \theta_{2i} p_i^{\beta}\} W_i} \right|. \end{aligned} \tag{5}$$

Integrating (2) between the limits $q_{2i}(t) = W_i$ to 0 when $t = t_{1i}$ to $t = t_{2i}$ we get

$$\begin{aligned} t_{2i} - t_{1i} &= p_i^{\beta} \int_0^{W_i} \frac{dq_{2i}(t)}{(\gamma + \theta_{2i} p_i^{\beta}) q_{2i}(t) + \alpha_i} \\ \Rightarrow t_{2i} &= t_{1i} + \frac{1}{K_{1i}} \ln \left| \frac{K_{1i} W_i + K_{2i}}{K_{2i}} \right|. \end{aligned} \tag{6}$$

5.1 Mathematical Formulation with allowable shortage and no inflation

The holding cost $(C_{H_i}^{SW_j})$ for the ith item in the jth secondary warehouse (SW_j) is given by

$$\begin{aligned} C_{H_i}^{SW_j} &= C_{1i}^{SW_j} \int_0^{t_{1i}} q_{1i}(t)\, dt \\ &= C_{1i}^{SW_j} \int_0^{t_{1i}} \frac{(\gamma + \theta_{2i} p_i^{\beta}) W_i + \alpha_i}{\theta_{1i} p_i^{\beta}} \{ e^{\theta_{1i}(t_{1i} - t)} - 1 \} dt \\ &= \frac{C_{1i}^{SW_j} K_{3i}}{\theta_{1i}} \left[(e^{\theta_{1i} t_{1i}} - 1) - \theta_{1i} t_{1i} \right]. \end{aligned} \tag{7}$$

The holding cost ($C_{H_i}^{PW}$) for the ith item in the primary warehouse (PW) is given by

$$\begin{aligned} C_{H_i}^{PW} &= C_{1i}^{PW}\left[\int_0^{t_{1i}} W_i\,dt + \int_{t_{1i}}^{t_{2i}} q_{2i}(t)\,dt\right] \\ &= C_{1i}^{PW}\left[W_i t_{1i} + \int_{t_{1i}}^{t_{2i}} \frac{K_{2i}}{K_{1i}}\{e^{K_{1i}(t_{2i}-t)} - 1\}\,dt\right] \\ &= C_{1i}^{PW}\left[W_i t_{1i} + \frac{K_{2i}}{K_{1i}^2}\left\{\left(e^{K_{1i}(t_{2i}-t_{1i})} - 1\right) - K_{1i}(t_{2i} - t_{1i})\right\}\right]. \end{aligned} \tag{8}$$

The total units (S_{RP_i}) of ith item transferred from SW_j to PW is given by

$$\begin{aligned} S_{RP_i} &= (S_i - W_i) - \int_0^{t_{1i}} \theta_{1i} q_{1i}(t)\,dt \\ &= (S_i - W_i) - \frac{\theta_{1i} C_{H_i}^{SW_j}}{C_{1i}^{SW_j}}. \end{aligned} \tag{9}$$

The Shortage cost (C_{S_i}) during the time period (t_{2i}, T_i) is given by

$$\begin{aligned} C_{S_i} &= -C_{2i}\int_{t_{2i}}^{T_i} q_{2i}(t)\,dt \\ &= -C_{2i}\int_{t_{2i}}^{T_i} \frac{K_{2i}}{\lambda}\left\{e^{-\lambda(T_i-t_{2i})} - e^{-\lambda(T_i-t)}\right\}dt \\ &= -C_{2i}\frac{K_{2i}}{\lambda^2}\left[\lambda(T_i - t_{2i})e^{-\lambda(T_i-t_{2i})} + \{e^{-\lambda(T_i-t_{2i})} - 1\}\right]. \end{aligned} \tag{10}$$

The opportunity cost (C_{OP_i}) during the time period (t_{2i}, T_i) is given by

$$\begin{aligned} C_{OP_i} &= C_{oi}\int_{t_{2i}}^{T_i}\left(D_i t - q_{2i}(t)\right)dt \\ &= C_{oi}\int_{t_{2i}}^{T_i}\left(f(p_i, 0)t - q_{2i}(t)\right)dt \\ &= C_{oi}\int_{t_{2i}}^{T_i}\left[\frac{\alpha_i}{p_i^\beta}t - \frac{K_{2i}}{\lambda}\left\{e^{-\lambda(T_i-t_{2i})} - e^{-\lambda(T_i-t)}\right\}dt\right. \\ &= C_{oi}\left[\frac{\alpha_i}{2p_i^\beta}(T_i^2 - t_{2i}^2) - \frac{K_{2i}}{\lambda^2}\left\{\lambda(T_i - t_{2i})e^{-\lambda(T_i-t_{2i})} + \left(e^{-\lambda(T_i-t_{2i})} - 1\right)\right\}\right]. \end{aligned} \tag{11}$$

Total Purchasing cost (P_{C_i}) for the ith item during the time period ($0,T_i$) is given by

$$P_{C_i} = C_i(S_i + R_i). \tag{12}$$

The total selling price (S_{p_i}) for the ith item during the time period ($0,T_i$) is given by

$$\begin{aligned} S_{p_i} &= p_i\int_0^{t_{2i}} D_i\,dt + p_i R_i \\ &= p_i\int_0^{t_{1i}} \frac{\alpha_i + \gamma W_i}{p_i^\beta}\,dt + p_i\int_{t_{1i}}^{t_{2i}} \frac{\alpha_i + \gamma q_{2i}(t)}{p_i^\beta}\,dt + p_i R_i \\ &= p_i\int_0^{t_{1i}} \frac{\alpha_i + \gamma W_i}{p_i^\beta}\,dt + p_i\int_{t_{1i}}^{t_{2i}} \frac{\alpha_i}{p_i^\beta}\,dt \end{aligned}$$

$$
\begin{aligned}
&+\frac{p_i\gamma}{p_i^{\beta}}\int_{t_{1i}}^{t_{2i}}\frac{K_{2i}}{K_{1i}}\left\{e^{K_{1i}(t_{2i}-t)}-1\right\}dt+p_iR_i\\
&=K_{4i}t_{1i}+p_iK_{2i}(t_{2i}-t_{1i})+\frac{K_{2i}p_i\gamma}{K_{1i}^2p_i^{\beta}}\left[\left\{e^{K_{1i}(t_{2i}-t_{1i})}-1\right\}\right.\\
&\left.-K_{1i}(t_{2i}-t_{1i})\right]+p_iR_i. \qquad (13)
\end{aligned}
$$

where

$$K_{4i}=\frac{p_i(\alpha_i+\gamma W_i)}{p_i^{\beta}}.$$

The transportation cost (TC_i^j) for transporting S_{RP_i} units from SW_j to PW is given by

$$TC_i^j=TC_i^{'}+S_{RP_i}.d_j.TC_i^{''},\quad j=1,2,\ldots,m.$$

where $TC_i^{'}$=Fixed transportation cost for the ith item (>0), and $TC_i^{''}$=Transportation cost per unit per unit distance (>0).

Let M_i be the period of permissible delay in settling account without extra charges.

Hence the interest payable per cycle and the interest earned per cycle for three cases be given as follows:

Case-I: $M_i\le t_{1i}<t_{2i}$.

Therefore the interest payable per cycle for the inventory not sold after the due date M_i is given by

$$
\begin{aligned}
P_T&=I_pC_i\left[\int_{M_i}^{t_{1i}}\left\{W_i+q_{1i}(t)\right\}dt+\int_{t_{1i}}^{t_{2i}}q_{2i}(t)\,dt+\int_{t_{2i}}^{T_i}q_{2i}(t)\,dt\right]\\
&=I_pC_i\left[\int_{M_i}^{t_{1i}}\left\{W_i+K_{3i}\left(e^{\theta_{1i}(t_{1i}-t)}-1\right)\right\}dt\right.\\
&\quad\left.+\int_{t_{1i}}^{t_{2i}}\frac{K_{2i}}{K_{1i}}\left\{e^{K_{1i}(t_{2i}-t)}-1\right\}dt+0\right],\ since\ q_{2i}(t)=0,\ \ for\ t_{2i}\le t\le T_i\\
&=I_pC_i\left[W_i\left(t_{1i}-M_i\right)+K_{3i}\left\{\frac{1}{\theta_{1i}}\left(e^{\theta_{1i}(t_{1i}-M_i)}-1\right)-\left(t_{1i}-M_i\right)\right\}\right.\\
&\quad\left.+\frac{K_{2i}}{K_{1i}^2}\left\{\left(e^{K_{1i}(t_{2i}-t_{1i})}-1\right)-K_{1i}\left(t_{2i}-t_{1i}\right)\right\}\right]. \qquad (14)
\end{aligned}
$$

The interest earned at time t during the positive inventory is given by

$$
\begin{aligned}
I_T&=p_iI_e\int_0^{t_{2i}}D_it\,dt\\
&=p_iI_e\left[\int_0^{t_{1i}}D_it\,dt+\int_{t_{1i}}^{t_{2i}}D_it\,dt\right]\\
&=p_iI_e\left[\int_0^{t_{1i}}f(p_i,W_i)t\,dt+\int_{t_{1i}}^{t_{2i}}f(p_i,q_i(t))t\,dt\right]\\
&=p_iI_e\left[\int_0^{t_{1i}}\frac{\alpha_i+\gamma W_i}{p_i^{\beta}}t\,dt+\int_{t_{1i}}^{t_{2i}}\frac{\alpha_i+\gamma q_{2i}(t)}{p_i^{\beta}}t\,dt\right]
\end{aligned}
$$

$$= p_i I_e \left[\frac{\alpha_i + \gamma W_i}{2 p_i^{\beta}} t_{1i}^2 + \frac{K_{2i}}{2} \left(t_{2i}^2 - t_{1i}^2 \right) - \frac{\gamma K_{2i}}{K_{1i}^3 p_i^{\beta}} \left\{ K_{1i} \left(t_{2i} - t_{1i} e^{K_{1i}(t_{2i} - t_{1i})} \right) + \left(1 - e^{K_{1i}(t_{2i} - t_{1i})} \right) + \frac{K_{1i}^2}{2} \left(t_{2i}^2 - t_{1i}^2 \right) \right\} \right]. \tag{15}$$

The profit function for the ith item is given by

$$PROF_i = Average\ profit\ for\ the\ ith\ item\ during\ (0, T_i)\ is\ given\ by$$
$$= \frac{1}{T_i} \left[S_{P_i} + I_T - C_{3i} - P_{C_i} - C_{H_i}^{SW_j} - C_{H_i}^{PW} - C_{S_i} - C_{OP_i} - P_T - TC_i^j \right].$$

Total average profit for N items is

$$PROF_I = \sum_{i=1}^{N} PROF_i \,.$$

Case-II: $t_{1i} \le M_i < t_{2i}$.

The profit function for the ith item is given by

$$PROF_i = Average\ profit\ for\ the\ ith\ item\ during\ (0, T_i)\ is\ given\ by$$
$$= \frac{1}{T_i} \left[S_{P_i} + I_T - C_{3i} - P_{C_i} - C_{H_i}^{SW_j} - C_{H_i}^{PW} - C_{S_i} - C_{OP_i} - P_T - TC_i^j \right].$$

where P_T and I_T are given by the expressions derived in section "Calculations for Case-II and Case-III with allowable shortage and no inflation" of Appendix [see Eqs. (25), (26) respectively].

Total average profit for N items is

$$PROF_{II} = \sum_{i=1}^{N} PROF_i \,.$$

Case-III: $t_{2i} \le M_i < T_i$.

In this case there is no interest payable.

The profit function for the ith item is given by

$$PROF_i = Average\ profit\ for\ the\ ith\ item\ during\ (0, T_i)\ is\ given\ by$$
$$= \frac{1}{T_i} \left[S_{P_i} + I_T - C_{3i} - P_{C_i} - C_{H_i}^{SW_j} - C_{H_i}^{PW} - C_{S_i} - C_{OP_i} - TC_i^j \right].$$

where I_T is given by the expression derived in section "Calculations for Case-II and Case-III with allowable shortage and no inflation" of Appendix [see Eq. (27)].

Total average profit for N items is

$$PROF_{III} = \sum_{i=1}^{N} PROF_i \,.$$

5.2 Mathematical Formulation with allowable shortage and inflation

The present value of holding cost ($C_{H_i}^{fSW_j}$) for the ith item in the jth secondary warehouse (SW_j) is given by

$$
\begin{aligned}
C_{H_i}^{fSW_j} &= C_{1i}^{SW_j}\int_0^{t_{1i}} q_{1i}(t)e^{-Rt}\,dt \\
&= C_{1i}^{SW_j}\int_0^{t_{1i}} \frac{(\gamma+\theta_{2i}p_i^{\beta})W_i+\alpha_i}{\theta_{1i}p_i^{\beta}}\{e^{\theta_{1i}(t_{1i}-t)}-1\}e^{-Rt}\,dt \\
&= C_{1i}^{SW_j}K_{3i}\left[\frac{1}{\theta_{1i}+R}\left(e^{\theta_{1i}t_{1i}}-e^{-Rt_{1i}}\right)+\frac{1}{R}\left(e^{-Rt_{1i}}-1\right)\right]. \qquad (16)
\end{aligned}
$$

The present value of holding cost ($C_{H_i}^{fPW}$) for the ith item in the primary warehouse (PW) is given by

$$
\begin{aligned}
C_{H_i}^{fPW} &= C_{1i}^{PW}\left[\int_0^{t_{1i}} W_i e^{-Rt}\,dt+\int_{t_{1i}}^{t_{2i}} q_{2i}(t)e^{-Rt}\,dt\right] \\
&= C_{1i}^{PW}\left[\int_0^{t_{1i}} W_i e^{-Rt}\,dt+\int_{t_{1i}}^{t_{2i}} \frac{K_{2i}}{K_{1i}}\{e^{K_{1i}(t_{2i}-t)}-1\}e^{-Rt}\,dt\right] \\
&= C_{1i}^{PW}\left[\frac{W_i}{R}\left(1-e^{-Rt_{1i}}\right)+\frac{K_{2i}}{K_{1i}}\left\{\frac{1}{k_{1i}+R}\left(e^{K_{1i}t_{2i}-(k_{1i}+R)t_{1i}}\right.\right.\right. \\
&\qquad \left.\left.\left.-e^{-Rt_{2i}}\right)+\frac{1}{R}\left(e^{-Rt_{2i}}-e^{-Rt_{1i}}\right)\right\}\right]. \qquad (17)
\end{aligned}
$$

The total units (S_{RP_i}) of ith item transferred from SW_j to PW is given by

$$
\begin{aligned}
S_{RP_i} &= (S_i-W_i)-\int_0^{t_{1i}}\theta_{1i}q_{1i}(t)\,dt \\
&= (S_i-W_i)-\frac{\theta_{1i}C_{H_i}^{SW_j}}{C_{1i}^{SW_j}}. \qquad (18)
\end{aligned}
$$

The present value of Shortage cost ($C_{S_i}^{f}$) during the time period (t_{2i}, T_i) is given by

$$
\begin{aligned}
C_{S_i}^{f} &= -C_{2i}\int_{t_{2i}}^{T_i} q_{2i}(t)e^{-Rt}\,dt \\
&= -C_{2i}\int_{t_{2i}}^{T_i}\frac{K_{2i}}{\lambda}\left\{e^{-\lambda(T_i-t_{2i})}-e^{-\lambda(T_i-t)}\right\}e^{-Rt}\,dt \\
&= -C_{2i}\frac{K_{2i}}{\lambda}\left[\frac{1}{R}\left\{e^{-\lambda(T_i-t_{2i})-Rt_{2i}}-e^{-\lambda(T_i-t_{2i})-RT_i}\right\}\right. \\
&\qquad \left.-\frac{1}{\lambda-R}\left\{e^{-RT_i}-e^{-\lambda T_i+(\lambda-R)t_{2i}}\right\}\right]. \qquad (19)
\end{aligned}
$$

The present value of opportunity cost (C_{OP_i}) during the time period (t_{2i}, T_i) is given by

$$
\begin{aligned}
C_{OP_i}^{f} &= C_{oi}\int_{t_{2i}}^{T_i}\left(D_i t-q_{2i}(t)\right)e^{-Rt}\,dt \\
&= C_{oi}\int_{t_{2i}}^{T_i}\left\{f(p_i,0)t-q_{2i}(t)\right\}e^{-Rt}\,dt \\
&= C_{oi}\int_{t_{2i}}^{T_i}\left[\frac{\alpha_i}{p_i^{\beta}}t-\frac{K_{2i}}{\lambda}\left\{e^{-\lambda(T_i-t_{2i})}-e^{-\lambda(T_i-t)}\right\}\right]e^{-Rt}\,dt
\end{aligned}
$$

$$= C_{oi}\Bigg[\frac{\alpha_i}{p_i^{\beta}}\bigg\{\frac{1}{R}\bigg(t_{2i}e^{-Rt_{2i}} - T_i e^{-RT_i}\bigg) + \frac{1}{R^2}\bigg(e^{-Rt_{2i}} - e^{-RT_i}\bigg)\bigg\}$$
$$-\frac{K_{2i}}{\lambda}\bigg\{\frac{1}{R}\bigg(e^{-\lambda(T_i-t_{2i})-Rt_{2i}} - e^{-\lambda(T_i-t_{2i})-RT_i}\bigg)$$
$$-\frac{1}{\lambda - R}\bigg(e^{-RT_i} - e^{-\lambda T_i+(\lambda-R)t_{2i}}\bigg)\bigg\}\Bigg]. \tag{20}$$

The present value of total Purchasing cost ($P_{C_i}^f$) for the ith item during the time period $(0,T_i)$ is given by

$$P_{C_i}^f = C_i\bigg(S_i + R_i e^{-RT_i}\bigg). \tag{21}$$

The present value of total selling price ($S_{p_i}^f$) for the ith item during the time period $(0,T_i)$ is given by

$$S_{p_i}^f = p_i\int_0^{t_{2i}} D_i e^{-Rt}\,dt + p_i R_i e^{-RT_i}$$
$$= p_i\int_0^{t_{1i}} \frac{\alpha_i + \gamma W_i}{p_i^{\beta}} e^{-Rt}\,dt + p_i\int_{t_{1i}}^{t_{2i}} \frac{\alpha_i + \gamma q_{2i}(t)}{p_i^{\beta}} e^{-Rt}\,dt + p_i R_i e^{-RT_i}$$
$$= p_i\int_0^{t_{1i}} \frac{\alpha_i + \gamma W_i}{p_i^{\beta}} e^{-Rt}\,dt + p_i\int_{t_{1i}}^{t_{2i}} \frac{\alpha_i}{p_i^{\beta}} e^{-Rt}\,dt$$
$$+\frac{p_i\gamma}{p_i^{\beta}}\int_{t_{1i}}^{t_{2i}} \frac{K_{2i}}{K_{1i}}\bigg\{e^{K_{1i}(t_{2i}-t)} - 1\bigg\}e^{-Rt}\,dt + p_i R_i e^{-RT_i}$$
$$= \frac{K_{4i}}{R}\bigg(1 - e^{-Rt_{1i}}\bigg) + \frac{p_i\alpha_i}{Rp_i^{\beta}}\bigg(e^{-Rt_{1i}} - e^{-Rt_{2i}}\bigg)$$
$$+\frac{K_{2i}p_i\gamma}{K_{1i}p_i^{\beta}}\bigg[\frac{1}{k_{1i}+R}\bigg\{e^{K_{1i}t_{2i}-(k_{1i}+R)t_{1i}} - e^{-Rt_{2i}}\bigg\}$$
$$+\frac{1}{R}\bigg(e^{-Rt_{2i}} - e^{-Rt_{1i}}\bigg)\bigg] + p_i R_i e^{-RT_i}. \tag{22}$$

where

$$K_{4i} = \frac{p_i(\alpha_i + \gamma W_i)}{p_i^{\beta}}.$$

The transportation cost (TC_i^j) for transporting S_{RP_i} units from SW_j to PW is given by

$$TC_i^j = TC_i^{'} + S_{RP_i}.d_j.TC_i^{''}, \quad j = 1, 2, \ldots, m.$$

where $TC_i^{'}$=Fixed transportation cost for the ith item (>0),and $TC_i^{''}$=Transportation cost per unit per unit distance (>0).

Case-I: $M_i \le t_{1i} < t_{2i}$.

Therefore the interest payable rate at time t is $(e^{i_p t} - 1)$ dollars per dollar, so the present value (at t=0) of interest payable rate at time t is $I_p(t) = (e^{i_p t} - 1)e^{-rt}$ dollars per dollar. Therefore, the interest payable per cycle for the inventory not sold after the due date M_i is given by

$$
\begin{aligned}
P_T^f &= C_i\Big[\int_{M_i}^{t_{1i}}\Big\{W_i+q_{1i}(t)\Big\}I_p(t)\,dt+\int_{t_{1i}}^{t_{2i}}q_{2i}(t)I_p(t)\,dt+\int_{t_{2i}}^{T_i}q_{2i}(t)I_p(t)\,dt\Big]\\
&= C_i\Big[\int_{M_i}^{t_{1i}}\Big\{W_i+K_{3i}\Big(e^{\theta_{1i}(t_{1i}-t)}-1\Big)\Big\}(e^{i_p t}-1)e^{-rt}\,dt\\
&\quad+\int_{t_{1i}}^{t_{2i}}\frac{K_{2i}}{K_{1i}}\Big\{e^{K_{1i}(t_{2i}-t)}-1\Big\}(e^{i_p t}-1)e^{-rt}\,dt+0\Big],\\
&\quad since\ q_{2i}(t)=0,\ \ for\ t_{2i}\le t\le T_i\\
&= C_i\Big[\frac{W_i}{l_p}\Big(e^{l_p t_{1i}}-e^{l_p M_i}\Big)+K_{3i}\Big\{\frac{1}{l_p-\theta_{1i}}\Big(e^{l_p t_{1i}}-e^{\theta_{1i}t_{1i}+(l_p-\theta_{1i})M_i}\Big)-\frac{1}{l_p}\Big(e^{l_p t_{1i}}-e^{l_p M_i}\Big)\Big\}\\
&\quad+\frac{W_i}{r}\Big(e^{-rt_{1i}}-e^{-rM_i}\Big)+K_{3i}\Big\{\frac{1}{\theta_{1i}+r}\Big(e^{-rt_{1i}}-e^{\theta_{1i}t_{1i}-(\theta_{1i}+r)M_i}\Big)-\frac{1}{r}\Big(e^{-rt_{1i}}-e^{-rM_i}\Big)\Big\}\\
&\quad+\frac{K_{2i}}{K_{1i}}\Big\{\frac{1}{l_p-K_{1i}}\Big(e^{l_p t_{2i}}-e^{K_{1i}t_{2i}+(l_p-K_{1i})t_{1i}}\Big)-\frac{1}{l_p}\Big(e^{l_p t_{2i}}-e^{l_p t_{1i}}\Big)\\
&\quad+\frac{1}{K_{1i}+r}\Big(e^{-rt_{2i}}-e^{K_{1i}t_{2i}-(K_{1i}+r)t_{1i}}\Big)-\frac{1}{r}\Big(e^{-rt_{2i}}-e^{-rt_{1i}}\Big)\Big\}\Big].
\end{aligned}
\tag{23}
$$

The present value of the interest earned at time t, $I_e(t)$ is $(e^{i_e t}-1)e^{-rt}$. The interest earned during the positive inventory, is given by

$$
\begin{aligned}
I_T^f &= p_i\int_0^{t_{2i}}D_i t(e^{i_e t}-1)e^{-rt}\,dt\\
&= p_i\Big[\int_0^{t_{1i}}D_i t(e^{i_e t}-1)e^{-rt}\,dt+\int_{t_{1i}}^{t_{2i}}D_i t(e^{i_e t}-1)e^{-rt}\,dt\Big]\\
&= p_i\Big[\int_0^{t_{1i}}f(p_i,W_i)t(e^{l_e t}-e^{-rt})\,dt+\int_{t_{1i}}^{t_{2i}}f(p_i,q_i(t))t(e^{l_e t}-e^{-rt})\,dt\Big]\\
&= p_i\Big[\int_0^{t_{1i}}\frac{\alpha_i+\gamma W_i}{p_i^\beta}t(e^{l_e t}-e^{-rt})\,dt+\int_{t_{1i}}^{t_{2i}}\frac{\alpha_i+\gamma q_{2i}(t)}{p_i^\beta}t(e^{l_e t}-e^{-rt})\,dt\Big]\\
&= p_i\Big[\frac{\alpha_i+\gamma W_i}{p_i^\beta}\Big\{\Big(\frac{t_{1i}e^{l_e t_{1i}}}{l_e}-\frac{e^{l_e t_{1i}}}{l_e^2}\Big)+\frac{1}{l_e^2}+\Big(\frac{t_{1i}e^{-rt_{1i}}}{r}+\frac{e^{-rt_{1i}}}{r^2}\Big)-\frac{1}{r^2}\Big\}\\
&\quad+\frac{\alpha_i}{p_i^\beta}\Big\{\Big(\frac{t_{2i}e^{l_e t_{2i}}}{l_e}-\frac{e^{l_e t_{2i}}}{l_e^2}\Big)-\Big(\frac{t_{1i}e^{l_e t_{1i}}}{l_e}-\frac{e^{l_e t_{1i}}}{l_e^2}\Big)+\Big(\frac{t_{2i}e^{-rt_{2i}}}{r}+\frac{e^{-rt_{2i}}}{r^2}\Big)\\
&\quad-\Big(\frac{t_{1i}e^{-rt_{1i}}}{r}+\frac{e^{-rt_{1i}}}{r^2}\Big)\Big\}+\frac{\gamma K_{2i}}{p_i^\beta K_{1i}}\Big\{\frac{t_{2i}e^{l_e t_{2i}}-t_{1i}e^{K_{1i}t_{2i}+(l_e-K_{1i})t_{1i}}}{l_e-K_{1i}}\\
&\quad-\frac{e^{l_e t_{2i}}-e^{K_{1i}t_{2i}+(l_e-K_{1i})t_{1i}}}{(l_e-K_{1i})^2}-\frac{t_{2i}e^{l_e t_{2i}}-t_{1i}e^{l_e t_{1i}}}{l_e}+\frac{e^{l_e t_{2i}}-e^{l_e t_{1i}}}{l_e^2}\\
&\quad+\frac{t_{2i}e^{-rt_{2i}}-t_{1i}e^{K_{1i}t_{2i}-(K_{1i}+r)t_{1i}}}{K_{1i}+r}+\frac{e^{-rt_{2i}}-e^{K_{1i}t_{2i}-(K_{1i}+r)t_{1i}}}{(K_{1i}+r)^2}\\
&\quad-\frac{t_{2i}e^{-rt_{2i}}-t_{1i}e^{-rt_{1i}}}{r}-\frac{e^{-rt_{2i}}-e^{-rt_{1i}}}{r^2}\Big\}\Big].
\end{aligned}
\tag{24}
$$

The profit function for the ith item is given by

$$
\begin{aligned}
PROF_i^f &= Average\ profit\ for\ the\ ith\ item\ during\ (0,T_i)\ is\ given\ by\\
&= \frac{1}{T_i}\Big[S_{P_i}^f+I_T^f-C_{3i}-P_{C_i}^f-C_{H_i}^{fSW_j}-C_{H_i}^{fPW}-C_{S_i}^f-C_{OP_i}^f-P_T^f-TC_i^j\Big].
\end{aligned}
$$

Total average profit for N items is

$$PROF_I^f = \sum_{i=1}^{N} PROF_i^f .$$

Case-II: $t_{1i} \le M_i < t_{2i}$.
The profit function for the ith item is given by

$$\begin{aligned} PROF_i^f &= Average\ profit\ for\ the\ ith\ item\ during\ (0, T_i)\ is\ given\ by \\ &= \frac{1}{T_i}\left[S_{P_i}^f + I_T^f - C_{3i} - P_{C_i}^f - C_{H_i}^{fSW_j} - C_{H_i}^{fPW} - C_{S_i}^f - C_{OP_i}^f - P_T^f - TC_i^j\right]. \end{aligned}$$

where P_T^f and I_T^f are given by the expressions derived in section "Calculations for Case-II and Case-III with allowable shortage and inflation" of Appendix [see Eqs. (28), (29) respectively].

Total average profit for N items is

$$PROF_{II}^f = \sum_{i=1}^{N} PROF_i^f .$$

Case-III: $t_{2i} \le M_i < T_i$.
In this case there is no interest payable.
The profit function for the ith item is given by

$$\begin{aligned} PROF_i^f &= Average\ profit\ for\ the\ ith\ item\ during\ (0, T_i)\ is\ given\ by \\ &= \frac{1}{T_i}\left[S_{P_i}^f + I_T^f - C_{3i} - P_{C_i}^f - C_{H_i}^{fSW_j} - C_{H_i}^{fPW} - C_{S_i}^f - C_{OP_i}^f - TC_i^j\right]. \end{aligned}$$

where I_T^f is given by the expression derived in section "Calculations for Case-II and Case-III with allowable shortage and inflation" Appendix 10.2 [see Eq. (30)].

Total average profit for N items is

$$PROF_{III}^f = \sum_{i=1}^{N} PROF_i^f .$$

6 Problem formulation

6.1 Model-1: Model with allowable shortage and no inflation

So, the above problem can be formulate as,

$$\begin{aligned} &\text{Maximize} \quad PROF_j \quad \text{j=I, II, III.} \\ &\text{subject to } : t_{1i} > 0, t_{2i} > t_{1i}, T_i > t_{2i}. \end{aligned}$$

6.2 Model-2: Model with allowable shortage and inflation

So, the above problem can be formulate as,

$$\begin{aligned} &\text{Maximize} \quad PROF_j^f \quad \text{j=I, II, III.} \\ &\text{subject to } : t_{1i} > 0, t_{2i} > t_{1i}, T_i > t_{2i}. \end{aligned}$$

7 Solution procedure

The above models are solved by using GA discussed in Sect. 2, and also by Particle Swarm Optimization (PSO) presented in Sect. 3. The parameters and steps of GA are as follows:

Population size=50, probability of crossover p_c=0.2, probability of mutation p_m =0.2, maximum generation=50. Let number of items=4.

Steps for solution:

1. Chromosomes with $T = (T_1, T_2, T_3, T_4)$ are represented(following 2(a)).
2. T_1, T_2, T_3, T_4 are randomly generated(following 2(b)).
3. $PROF_i$ are evaluated(following 2(c)).
4. Roulette wheel selection process is used(following 2(d)).
5. The solution is taken for crossover and crossover taken place on the selected solutions(following 2(e)).
6. The selection is taken for mutation(following 2(f)).

After the application of selection, crossover and mutation, the new population is ready for its next iteration.

8 Numerical illustration

8.1 Model-1: with allowable shortage and no inflation

An example is presented to illustrate the effect of the inventory model developed here with the following numerical data and the data for four items are shown in Table 1. $\mu = 0.9$, $I_e = 0.12$, $I_p = 0.15$, $d_1 = 1.5$, $d_2 = 1.6$, $\lambda = 0.005$, $\beta = 0.18$, $\gamma = 0.75$

According to the developed GA and PSO for the proposed inventory system, the optimal solution has been obtained for different cases. The optimal values of M_i and T_i (i=1,2,3,4) along with the average profit are displayed in Table 2.

Table 1 Some data for model-1

Item	C_{1i}^{PW}	C_{2i}	C_{oi}	C_{3i}	C_i	p_i	S_i	W_i	θ_{1i}	θ_{2i}	TC_i'	TC_i''	α_i
Item-1	0.784	0.93	0.31	72.0	3.5	6.2	80.0	15.0	0.010	0.013	0.64	0.010	100.0
Item-2	0.782	0.95	0.33	74.0	3.6	6.4	85.0	15.0	0.011	0.012	0.65	0.011	112.0
Item-3	0.781	0.90	0.35	76.0	4.2	6.6	87.0	15.0	0.013	0.014	0.66	0.012	113.0
Item-4	0.785	0.97	0.37	75.0	4.4	6.5	90.0	15.0	0.012	0.015	0.67	0.013	115.0

Table 2 Optimal solutions of Model-1 for three different cases using GA and PSO

Case	M_1	M_2	M_3	M_4		T_1	T_2	T_3	T_4	Total average profit
Case-I	0.31	0.32	0.33	0.35	GA	1.84	1.73	1.78	1.73	278.70
					PSO	1.73	1.69	1.74	1.70	277.88
Case-II	0.87	0.84	0.85	0.86	GA	1.69	1.69	1.69	1.69	303.25
					PSO	1.71	1.70	1.71	1.70	302.82
Case-III	1.21	1.22	1.23	1.24	GA	1.69	1.58	1.680	1.60	342.10
					PSO	1.73	1.67	1.71	1.62	341.53

Table 3 Data for Model-2

Item	C_{1i}^{PW}	C_{2i}	C_{oi}	C_{3i}	C_i	p_i	S_i	W_i	θ_{1i}	θ_{2i}	TC_i'	TC_i''	α_i
Item-1	0.784	0.93	0.31	72.0	3.5	6.2	80.0	15.0	0.010	0.013	0.64	0.010	100.0
Item-2	0.782	0.95	0.33	74.0	3.6	6.4	85.0	15.0	0.011	0.012	0.65	0.011	112.0
Item-3	0.781	0.90	0.35	76.0	4.2	6.6	87.0	15.0	0.013	0.014	0.66	0.012	113.0
Item-4	0.785	0.97	0.37	75.0	4.4	6.5	90.0	15.0	0.012	0.015	0.67	0.013	115.0

Table 4 Optimal solutions of Model-2 for three different cases using GA and PSO

Case	M_1	M_2	M_3	M_4		T_1	T_2	T_3	T_4	Total average profit
Case-I	0.31	0.32	0.33	0.35	GA	1.69	1.62	1.69	1.60	266.76
					PSO	1.79	1.75	1.68	1.68	265.76
Case-II	0.87	0.84	0.85	0.86	GA	1.68	1.58	1.65	1.60	280.24
					PSO	1.78	1.72	1.64	1.61	279.33
Case-III	1.21	1.22	1.23	1.24	GA	1.66	1.58	1.68	1.60	287.03
					PSO	1.72	1.70	1.71	1.65	285.99

8.2 Model-2: with allowable shortage and inflation

The following numerical data and the data for four items in Table 3 are considered. μ=0.9, i_e =0.12, i_p=0.15,d_1=1.5, d_2=1.6,λ=0.005, β=0.18, γ=0.75, r=0.1, f=0.05, l_p=0.05, l_e=0.02

The optimum results of Model-2 with inflation are shown in the following Table 4 for different M_i

From the Tables 2 and 4, it is observed that the Model-1 without inflation fetches more profits than the Model-2 with inflation in all cases. Again, from each table, it is concluded that case-III gives the highest profit and case-I the least profit. This is because the delayed period of payment is maximum in case-III and minimum in case-I (cf. Figs. 2,3,4). Hence, the behavior of the optimum results are as per our expectation.

Sensitivity analysis for Model-1 (without inflation model):

For the given numerical example, sensitivity analysis is performed using GA to study the effect of changes in the different parameters (β and γ) of demand on the cycle length and total average profit(TAP). The optimum results are shown in Table 5 and Figs. 5 and 6. It is seen that when β increases, demand decreases and as such profit decreases; also when γ increases, demand increases and as such profit increases; which agrees with reality.

Sensitivity analysis for Model-2 (with inflation model):

Sensitivity analyses are also performed using GA for Model-2 w.r.to different β and γ and optimum total average profits(TAP) are presented in Figs. 7 and 8 respectively, when other input values are same. The optimum values of T_1, T_2, T_3 and $T_4$are obtained for all cases and these values are as usual like the values in Table 5.

In these studies, the behavior of total average profit(TAP) with respect to different cases and models are same as the original studies presented in Tables 2 and 4.

Sensitivity analysis w.r.to the parameter of GA

For the given numerical data, sensitivity analysis is performed to study the effect of changes in the parameter p_c of GA on the total average profit. The optimum results of Model-1 and 2 are shown in Table 6. It is seen that when p_c increases total average profit increases slightly.

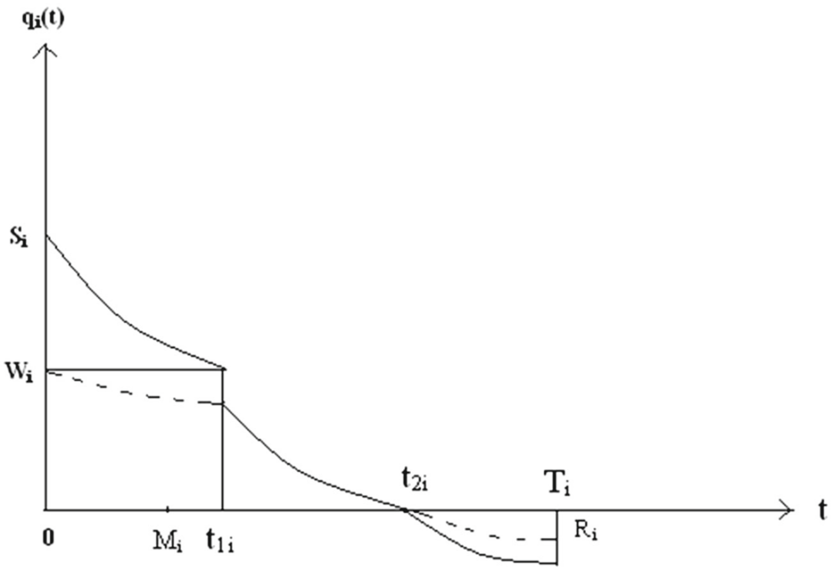

Fig. 2 Pictorial representation of the inventory behavior in a multi-warehouse system for deteriorating items of case I

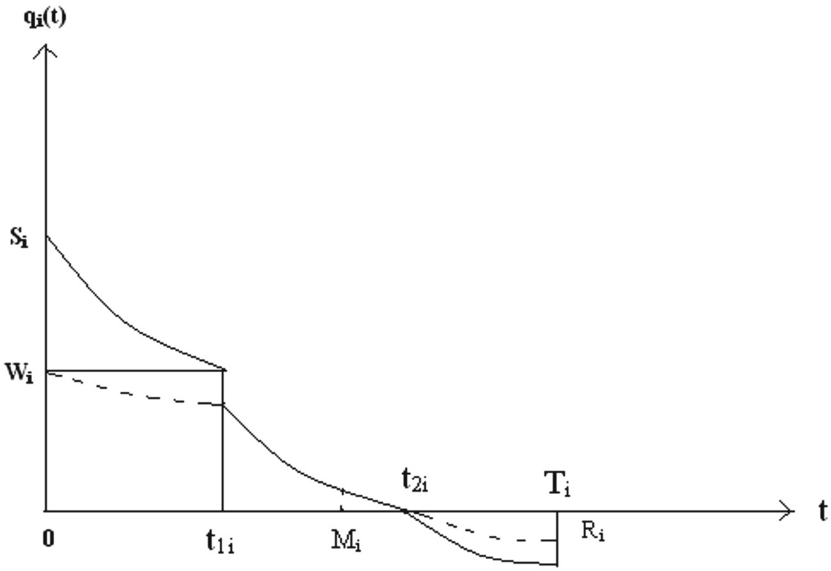

Fig. 3 Pictorial representation of the inventory behavior in a multi-warehouse system for deteriorating items of case II

Comparison of results using GA and PSO:

It is observed that in all cases GA gives the better results than particle swarm optimization (PSO). Also it is observed that in GA after fifty iterations we get the above results but in PSO we get the results by taking more than fifty iterations. For comparison we consider first 60 runs of both the algorithms and perform a t-test to study the convergence. The result of t-test

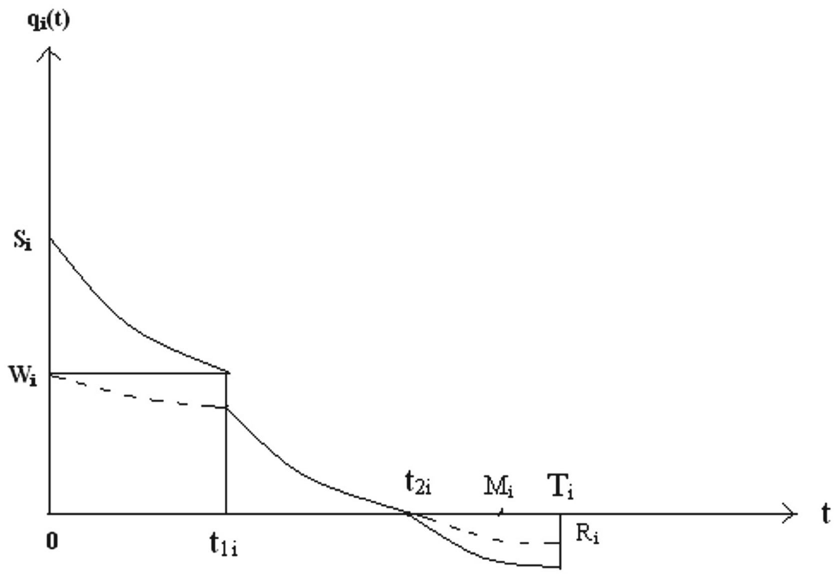

Fig. 4 Pictorial representation of the inventory behavior in a multi-warehouse system for deteriorating items of case III

Table 5 The sensitivity analysis of β and γ for Model-1 due to different β when $\gamma = 0.75$ and different γ when $\beta = 0.18$

Case	$\beta = 0.16$	$\beta = 0.17$	$\beta = 0.18$	$\beta = 0.19$	$\beta = 0.20$	$\gamma = 0.55$	$\gamma = 0.65$	$\gamma = 0.75$	$\gamma = 0.85$	$\gamma = 0.95$
I										
T_1	1.69	1.84	1.84	1.85	1.88	1.96	1.87	1.84	1.69	1.69
T_2	1.65	1.69	1.73	1.74	1.75	1.85	1.80	1.73	1.69	1.64
T_3	1.70	1.69	1.78	1.78	1.80	1.90	1.80	1.78	1.69	1.65
T_4	1.69	1.69	1.73	1.74	1.78	1.90	1.77	1.73	1.69	1.60
TAP	303.34	290.93	278.70	266.64	254.51	212.95	252.09	278.70	298.16	313.09
II										
T_1	1.69	1.69	1.69	1.84	1.84	1.96	1.84	1.69	1.69	1.66
T_2	1.64	1.64	1.69	1.69	1.73	1.80	1.73	1.69	1.65	1.58
T_3	1.65	1.65	1.69	1.69	1.78	1.83	1.78	1.69	1.65	1.60
T_4	1.60	1.60	1.69	1.69	1.73	1.77	1.73	1.69	1.60	1.59
TAP	327.34	315.29	303.25	291.37	279.55	236.50	276.23	303.25	323.03	338.01
III										
T_1	1.66	1.66	1.69	1.69	1.78	1.88	1.78	1.69	1.66	1.57
T_2	1.55	1.58	1.58	1.59	1.64	1.74	1.64	1.58	1.55	1.44
T_3	1.67	1.68	1.68	1.69	1.70	1.82	1.78	1.68	1.68	1.43
T_4	1.52	1.54	1.60	1.60	1.61	1.77	1.60	1.60	1.54	1.50
TAP	372.41	357.21	342.10	327.03	311.82	269.06	312.09	342.10	364.13	382.14

is shown in Table 7. Also the convergence graph of average fitness is shown in Fig. 9. In view of that the performance of GA is acceptable. Moreover, from Table 7 it is clear that there is no significant difference in the mean with the two optimization algoriths. In addition, PSO can also provide a more stable and reliable solution, because it yields significantly smaller standard deviation.

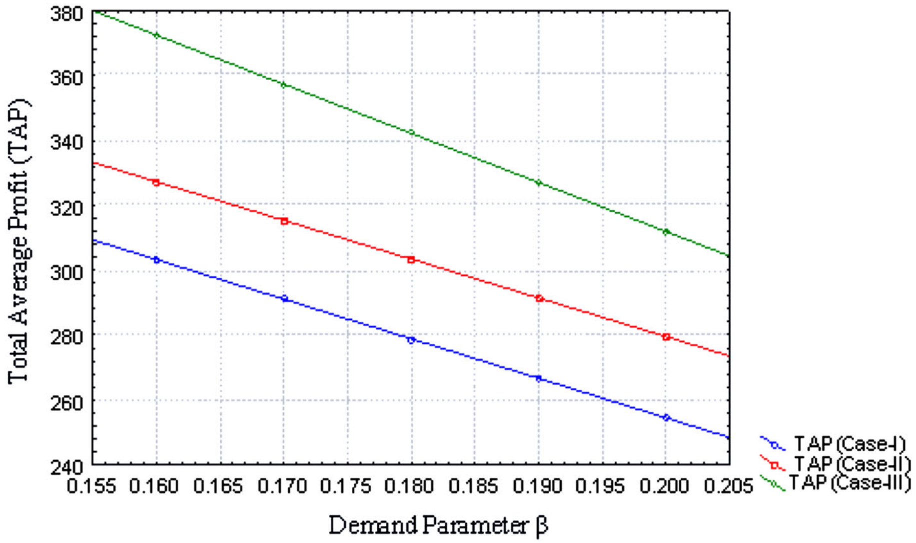

Fig. 5 Graph of sensitivity analysis for Model-1 due to different β when $\gamma = 0.75$

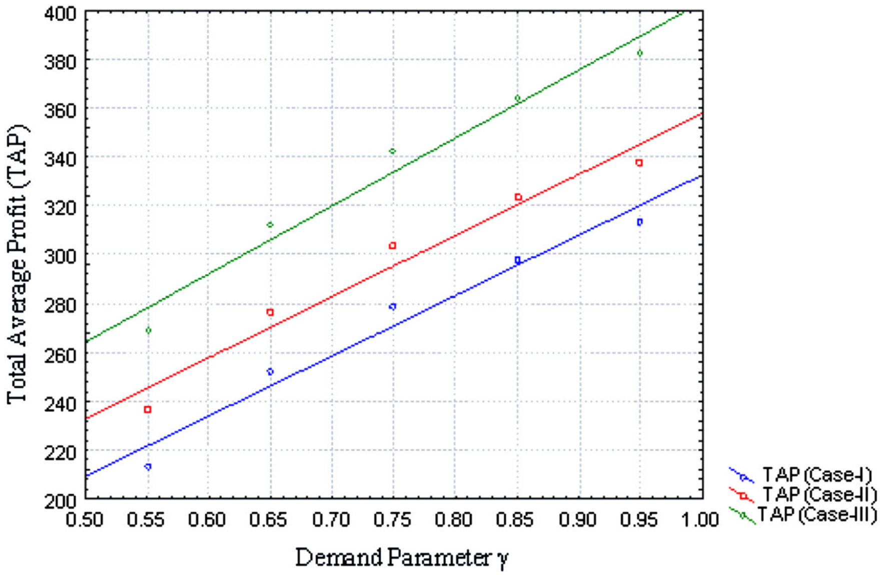

Fig. 6 Graph of sensitivity analysis for Model-1 due to different γ when $\beta = 0.18$

9 Conclusion

In this paper, a multi-storage inventory model for multi-item has been formulated and solved via contractive mapping genetic algorithm (CMGA)and PSO. The demand of the items is dependent on selling price and stock-level at the primary warehouse. In the model the stocks of SWs are transferred to PW under continuous release pattern and the associated transportation cost directly varies with the distance from PW to SWs but the holding cost of an item in SWs

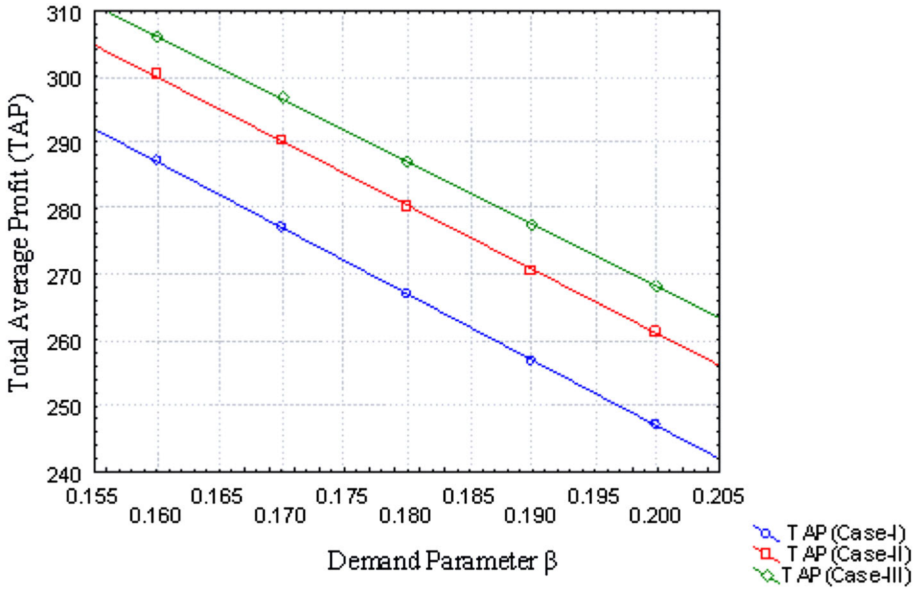

Fig. 7 Graph of sensitivity analysis for Model-2 due to different β when $\gamma = 0.75$

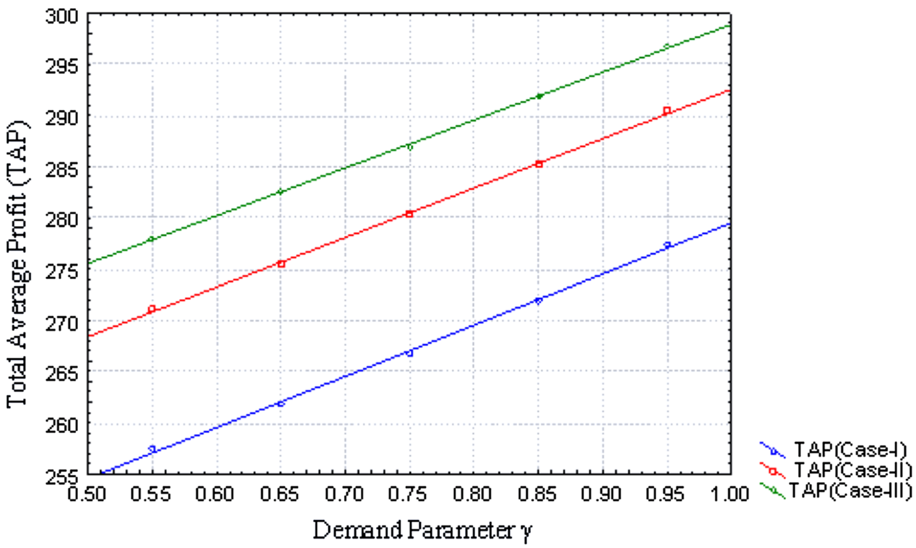

Fig. 8 Graph of sensitivity analysis for Model-2 due to different γ when $\beta = 0.18$

Table 6 Sensitivity analysis with respect to parameter p_c

p_c	Total average profit (case-I)		Total average profit (case-II)		Total average profit (case-III)	
	No inflation	Inflation	No inflation	Inflation	No inflation	Inflation
0.15	278.61	266.70	303.18	280.09	341.47	286.57
0.2	278.70	266.76	303.25	280.24	341.53	287.03
0.25	278.76	266.81	303.37	280.31	341.69	287.98

Table 7 Statistical analysis using t-test

Set		T_1	T_2	T_3	T_4	Mean	SD	N	DF	t-test value	Analysis
GEN-1	GA	1.43	1.74	1.97	1.54	1.6700	0.2376	4			
	PSO	1.82	1.74	1.72	1.75	1.7575	0.0435	4	6	0.7244	
GEN-3	GA	1.43	1.74	1.97	1.54	1.6700	0.2376	4			
	PSO	1.83	1.70	1.76	1.67	1.7450	0.0645	4	6	0.7244	
GEN-5	GA	1.43	1.74	1.97	1.54	1.6700	0.2376	4			
	PSO	1.83	1.70	1.76	1.67	1.7450	0.0645	4	6	0.7244	
GEN-7	GA	1.43	1.76	1.97	1.64	1.7000	0.2258	4			
	PSO	1.79	1.72	1.78	1.77	1.7650	0.0311	4	6	0.5703	
GEN-10	GA	1.43	1.74	1.97	1.54	1.6700	0.2376	4			
	PSO	1.80	1.69	1.77	1.68	1.7350	0.0592	4	6	0.5309	
GEN-12	GA	1.67	1.78	1.96	1.68	1.7725	0.1345	4			
	PSO	1.73	1.67	1.76	1.69	1.7125	0.0403	4	6	0.8546	
GEN-15	GA	1.72	1.77	1.96	1.94	1.8475	0.1204	4			
	PSO	1.73	1.69	1.74	1.70	1.7150	0.0238	4	6	2.1595	
GEN-17	GA	1.72	1.75	1.76	1.69	1.7300	0.0316	4			
	PSO	1.73	1.68	1.75	1.66	1.7050	0.0420	4	6	0.9506	
GEN-20	GA	1.72	1.77	1.96	1.71	1.7900	0.1163	4			
	PSO	1.75	1.75	1.72	1.65	1.7400	0.0173	4	6	0.7212	
GEN-22	GA	1.71	1.76	1.95	1.73	1.7875	0.1103	4			
	PSO	1.79	1.72	1.75	1.69	1.7375	0.0427	4	6	0.8457	
GEN-25	GA	1.72	1.77	1.96	1.71	1.7900	0.1163	4			
	PSO	1.83	1.69	1.77	1.72	1.7633	0.0702	4	6	0.3475	$T_{tab} = 2.447$, so we
GEN-27	GA	1.69	1.71	1.76	1.68	1.7100	0.0356	4			
	PSO	1.79	1.64	1.72	1.69	1.7100	0.0627	4	6	0.0000	fail to reject H_0
GEN-30	GA	1.72	1.77	1.66	1.71	1.7150	0.0451	4			
	PSO	1.81	1.79	1.74	1.68	1.7550	0.0580	4	6	1.0887	at the 5%
GEN-32	GA	1.71	1.69	1.65	1.68	1.6825	0.0250	4			
	PSO	1.73	1.68	1.62	1.69	1.6800	0.0455	4	6	0.0964	level of significance
GEN-35	GA	1.72	1.77	1.66	1.71	1.7150	0.0451	4			
	PSO	1.73	1.69	1.74	1.70	1.7150	0.0238	4	6	0.0000	
GEN-37	GA	1.71	1.69	1.65	1.68	1.6825	0.0250	4			
	PSO	1.72	1.66	1.71	1.72	1.7025	0.0287	4	6	1.0505	
GEN-40	GA	1.72	1.77	1.66	1.71	1.7150	0.0451	4			
	PSO	1.71	1.69	1.72	1.74	1.7150	0.0238	4	6	0.0000	
GEN-42	GA	1.82	1.75	1.69	1.72	1.7450	0.0557	4			
	PSO	1.76	1.72	1.73	1.67	1.7200	0.0374	4	6	0.7454	
GEN-45	GA	1.84	1.73	1.78	1.73	1.7700	0.0523	4			
	PSO	1.78	1.75	1.74	1.68	1.7375	0.0419	4	6	0.9699	
GEN-47	GA	1.83	1.71	1.76	1.73	1.7575	0.0525	4			
	PSO	1.82	1.74	1.76	1.69	1.7525	0.0538	4	6	0.1330	
GEN-50	GA	1.84	1.73	1.78	1.73	1.7700	0.0523	4			
	PSO	1.73	1.69	1.74	1.70	1.7150	0.0238	4	6	1.9149	
GEN-52	GA	1.83	1.72	1.77	1.72	1.7600	0.0523	4			
	PSO	1.81	1.71	1.75	1.69	1.7400	0.0529	4	6	0.5377	
GEN-55	GA	1.84	1.73	1.78	1.73	1.7700	0.0523	4			
	PSO	1.81	1.73	1.73	1.73	1.7500	0.0400	4	6	0.6076	
GEN-57	GA	1.84	1.73	1.78	1.73	1.7700	0.0523	4			
	PSO	1.83	1.72	1.74	1.67	1.7400	0.0668	4	6	0.7071	
GEN-60	GA	1.84	1.73	1.78	1.73	1.7700	0.0523	4			
	PSO	1.84	1.64	1.74	1.68	1.7250	0.0870	4	6	0.8868	

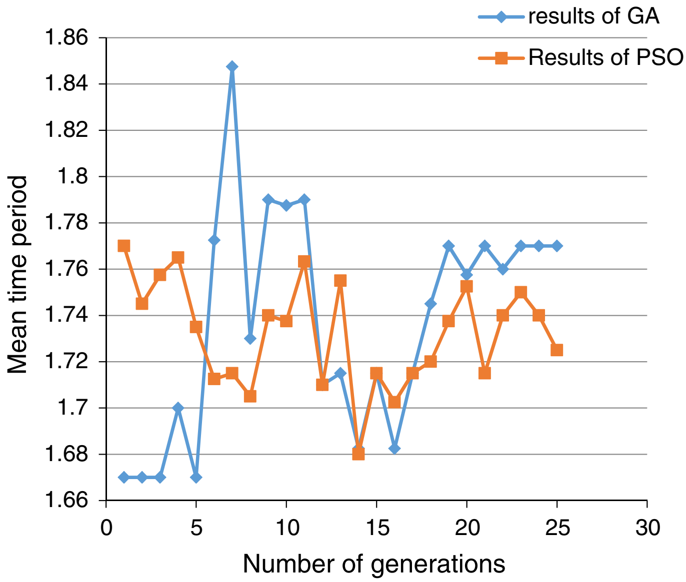

Fig. 9 Convergence graph of average fitness

has reverse effect with the distance. We have formulated the problem to optimize a general non-linear function and solve numerically by CMGA and PSO. Lastly, to study the effect of the decision variable on the changes of different parameters, a sensitivity analysis is also performed. For further research one can generalize this model to include the case of finite rate of replenishment and finite time horizon. The model can also be formulated and solved in a fuzzy or fuzzy-random environment.

10 Appendix

10.1 Calculations for Case-II and Case-III with allowable shortage and no inflation

Case-II: $(t_{1i} \le M_i < t_{2i})$.

The interest payable per cycle for the inventory not sold after the due date M_i is given by

$$\begin{aligned} P_T &= I_p C_i \left[\int_{M_i}^{t_{2i}} q_{2i}(t)\,dt + \int_{t_{2i}}^{T_i} q_{2i}(t)\,dt \right] \\ &= I_p C_i \int_{M_i}^{t_{2i}} \frac{K_{2i}}{K_{1i}} \left\{ e^{K_{1i}(t_{2i}-t)} - 1 \right\} dt + 0 \Big], \ since\ q_{2i}(t) = 0,\ \ for\ t_{2i} \le t \le T_i \\ &= I_p C_i \frac{K_{2i}}{K_{1i}^2} \left[\left\{ e^{K_{1i}(t_{2i}-M_i)} - 1 \right\} - K_{1i} \Big(t_{2i} - M_i \Big) \right]. \end{aligned} \tag{25}$$

The interest earned at time t during the positive inventory is given by

$$I_T = p_i I_e \int_0^{t_{2i}} D_i t\,dt$$

$$
\begin{aligned}
&= p_i I_e\left[\int_0^{t_{1i}} D_i t\,dt + \int_{t_{1i}}^{t_{2i}} D_i t\,dt\right]\\
&= p_i I_e\left[\int_0^{t_{1i}} f(p_i, W_i)t\,dt + \int_{t_{1i}}^{t_{2i}} f(p_i, q_i(t))t\,dt\right]\\
&= p_i I_e\left[\int_0^{t_{1i}} \frac{\alpha_i + \gamma W_i}{p_i^{\beta}} t\,dt + \int_{t_{1i}}^{t_{2i}} \frac{\alpha_i + \gamma q_{2i}(t)}{p_i^{\beta}} t\,dt\right]\\
&= p_i I_e\left[\frac{\alpha_i + \gamma W_i}{2p_i^{\beta}} t_{1i}^2 + \frac{K_{2i}}{2}\left(t_{2i}^2 - t_{1i}^2\right) - \frac{\gamma K_{2i}}{K_{1i}^3 p_i^{\beta}}\left\{K_{1i}\left(t_{2i} - t_{1i}e^{K_{1i}(t_{2i}-t_{1i})}\right)\right.\right.\\
&\qquad \left.\left. + \left(1 - e^{K_{1i}(t_{2i}-t_{1i})}\right) + \frac{K_{1i}^2}{2}\left(t_{2i}^2 - t_{1i}^2\right)\right\}\right].
\end{aligned} \tag{26}
$$

Case-III: ($t_{2i} \leq M_i < T_i$).

The interest earned at time t during the positive inventory period plus the interest earned from the cash invested during the time period (t_{2i}, M_i) after the inventory is exhausted at time t_{2i}, and it is given by

$$
\begin{aligned}
I_T &= p_i I_e\left[\int_0^{t_{2i}} D_i t\,dt + (M_i - t_{2i})\int_0^{t_{2i}} D_i\,dt\right]\\
&= p_i I_e\left[\int_0^{t_{1i}} D_i t\,dt + \int_{t_{1i}}^{t_{2i}} D_i t\,dt\right.\\
&\qquad \left. +(M_i - t_{2i})\left\{\int_0^{t_{1i}} \frac{\alpha_i + \gamma W_i}{p_i^{\beta}}\,dt + \int_{t_{1i}}^{t_{2i}} \frac{\alpha_i + \gamma q_{2i}(t)}{p_i^{\beta}}\,dt\right\}\right]\\
&= p_i I_e\left[\int_0^{t_{1i}} f(p_i, W_i)t\,dt + \int_{t_{1i}}^{t_{2i}} f(p_i, q_i(t))t\,dt + (M_i - t_{2i})\left\{\int_0^{t_{1i}} \frac{\alpha_i + \gamma W_i}{p_i^{\beta}}\,dt\right.\right.\\
&\qquad \left.\left. + \int_{t_{1i}}^{t_{2i}} \frac{\alpha_i}{p_i^{\beta}}\,dt + \frac{\gamma}{p_i^{\beta}}\int_{t_{1i}}^{t_{2i}} \frac{K_{2i}}{K_{1i}}\left(e^{K_{1i}(t_{2i}-t)} - 1\right)dt\right\}\right]\\
&= p_i I_e\left[\int_0^{t_{1i}} \frac{\alpha_i + \gamma W_i}{p_i^{\beta}} t\,dt + \int_{t_{1i}}^{t_{2i}} \frac{\alpha_i + \gamma q_{2i}(t)}{p_i^{\beta}} t\,dt\right] + (M_i - t_{2i}) I_e\left[K_{4i}t_{1i}\right.\\
&\qquad \left. + p_i K_{2i}(t_{2i} - t_{1i}) + \frac{K_{2i}p_i\gamma}{K_{1i}^2 p_i^{\beta}}\left\{\left(e^{K_{1i}(t_{2i}-t_{1i})} - 1\right) - K_{1i}(t_{2i} - t_{1i})\right\}\right]\\
&= p_i I_e\left[\frac{\alpha_i + \gamma W_i}{2p_i^{\beta}} t_{1i}^2 + \frac{K_{2i}}{2}\left(t_{2i}^2 - t_{1i}^2\right) - \frac{\gamma K_{2i}}{K_{1i}^3 p_i^{\beta}}\left\{K_{1i}\left(t_{2i} - t_{1i}e^{K_{1i}(t_{2i}-t_{1i})}\right)\right.\right.\\
&\qquad \left.\left. + \left(1 - e^{K_{1i}(t_{2i}-t_{1i})}\right) + \frac{K_{1i}^2}{2}\left(t_{2i}^2 - t_{1i}^2\right)\right\}\right] + (M_i - t_{2i}) I_e\left[K_{4i}t_{1i} + p_i K_{2i}(t_{2i} - t_{1i})\right.\\
&\qquad \left. + \frac{K_{2i}p_i\gamma}{K_{1i}^2 p_i^{\beta}}\left\{\left(e^{K_{1i}(t_{2i}-t_{1i})} - 1\right) - K_{1i}(t_{2i} - t_{1i})\right\}\right].
\end{aligned} \tag{27}
$$

10.2 Calculations for Case-II and Case-III with allowable shortage and inflation

Case-II: ($t_{1i} \leq M_i < t_{2i}$).

The interest payable rate at time t is $(e^{i_p t} - 1)$ dollars per dollar, so the present value (at t=0) of interest payable rate at time t is $I_p(t) = (e^{i_p t} - 1)e^{-rt}$dollars per dollar. Therefore, the interest payable per cycle for the inventory not sold after the due date M_i is given by

$$\begin{aligned}
P_T^f &= C_i\left[\int_{M_i}^{t_{2i}} q_{2i}(t)I_p(t)\,dt + \int_{t_{2i}}^{T_i} q_{2i}(t)I_p(t)\,dt\right] \\
&= C_i\left[\int_{M_i}^{t_{2i}} \frac{K_{2i}}{K_{1i}}\left\{e^{K_{1i}(t_{2i}-t)} - 1\right\}(e^{i_p t} - 1)e^{-rt}\,dt + 0\right], \\
&\quad \mathit{since}\ q_{2i}(t) = 0,\ \ \mathit{for}\ t_{2i} \le t \le T_i \\
&= C_i\frac{K_{2i}}{K_{1i}}\left[\frac{1}{l_p - K_{1i}}\left(e^{l_p t_{2i}} - e^{K_{1i}t_{2i}+(l_p-K_{1i})M_i}\right) - \frac{1}{l_p}\left(e^{l_p t_{2i}} - e^{l_p M_i}\right)\right. \\
&\quad \left. + \frac{1}{K_{1i}+r}\left(e^{-rt_{2i}} - e^{K_{1i}t_{2i}-(K_{1i}+r)M_i}\right) - \frac{1}{r}\left(e^{-rt_{2i}} - e^{-rM_i}\right)\right].
\end{aligned} \tag{28}$$

The present value of the interest earned at time t, $I_e(t)$ is $(e^{i_e t} - 1)e^{-rt}$. The interest earned during the positive inventory, is given by

$$\begin{aligned}
I_T^f &= p_i\int_0^{t_{2i}} D_i t(e^{i_e t} - 1)e^{-rt}\,dt \\
&= p_i\left[\int_0^{t_{1i}} D_i t(e^{i_e t} - 1)e^{-rt}\,dt + \int_{t_{1i}}^{t_{2i}} D_i t(e^{i_e t} - 1)e^{-rt}\,dt\right] \\
&= p_i\left[\int_0^{t_{1i}} f(p_i, W_i)t(e^{l_e t} - e^{-rt})\,dt + \int_{t_{1i}}^{t_{2i}} f(p_i, q_i(t))t(e^{l_e t} - e^{-rt})\,dt\right] \\
&= p_i\left[\int_0^{t_{1i}} \frac{\alpha_i + \gamma W_i}{p_i^\beta}t(e^{l_e t} - e^{-rt})\,dt + \int_{t_{1i}}^{t_{2i}} \frac{\alpha_i + \gamma q_{2i}(t)}{p_i^\beta}t(e^{l_e t} - e^{-rt})\,dt\right] \\
&= p_i\left[\frac{\alpha_i + \gamma W_i}{p_i^\beta}\left\{\left(\frac{t_{1i}e^{l_e t_{1i}}}{l_e} - \frac{e^{l_e t_{1i}}}{l_e^2}\right) + \frac{1}{l_e^2} + \left(\frac{t_{1i}e^{-rt_{1i}}}{r} + \frac{e^{-rt_{1i}}}{r^2}\right) - \frac{1}{r^2}\right\}\right. \\
&\quad + \frac{\alpha_i}{p_i^\beta}\left\{\left(\frac{t_{2i}e^{l_e t_{2i}}}{l_e} - \frac{e^{l_e t_{2i}}}{l_e^2}\right) - \left(\frac{t_{1i}e^{l_e t_{1i}}}{l_e} - \frac{e^{l_e t_{1i}}}{l_e^2}\right) + \left(\frac{t_{2i}e^{-rt_{2i}}}{r} + \frac{e^{-rt_{2i}}}{r^2}\right)\right. \\
&\quad \left. - \left(\frac{t_{1i}e^{-rt_{1i}}}{r} + \frac{e^{-rt_{1i}}}{r^2}\right)\right\} + \frac{\gamma K_{2i}}{p_i^\beta K_{1i}}\left\{\frac{t_{2i}e^{l_e t_{2i}} - t_{1i}e^{K_{1i}t_{2i}+(l_e-K_{1i})t_{1i}}}{l_e - K_{1i}}\right. \\
&\quad - \frac{e^{l_e t_{2i}} - e^{K_{1i}t_{2i}+(l_e-K_{1i})t_{1i}}}{(l_e - K_{1i})^2} - \frac{t_{2i}e^{l_e t_{2i}} - t_{1i}e^{l_e t_{1i}}}{l_e} + \frac{e^{l_e t_{2i}} - e^{l_e t_{1i}}}{l_e^2} \\
&\quad + \frac{t_{2i}e^{-rt_{2i}} - t_{1i}e^{K_{1i}t_{2i}-(K_{1i}+r)t_{1i}}}{K_{1i}+r} + \frac{e^{-rt_{2i}} - e^{K_{1i}t_{2i}-(K_{1i}+r)t_{1i}}}{(K_{1i}+r)^2} \\
&\quad \left.\left. - \frac{t_{2i}e^{-rt_{2i}} - t_{1i}e^{-rt_{1i}}}{r} - \frac{e^{-rt_{2i}} - e^{-rt_{1i}}}{r^2}\right\}\right].
\end{aligned} \tag{29}$$

Case-III: $(t_{2i} \le M_i < T_i)$.

The interest earned per cycle is the interest earned during the positive inventory period plus the interest earned from the cash invested during the time period (t_{2i}, M_i) after the inventory is exhausted at time t_{2i}, is given by

$$\begin{aligned}
I_T^f &= p_i\left[\int_0^{t_{2i}} D_i t(e^{i_e t} - 1)e^{-rt}\,dt + \left(e^{l_e(M_i-t_{2i})} - 1\right)\int_0^{t_{2i}} D_i\,dt\right] \\
&= p_i\left[\int_0^{t_{1i}} D_i t(e^{i_e t} - 1)e^{-rt}\,dt + \int_{t_{1i}}^{t_{2i}} D_i t(e^{i_e t} - 1)e^{-rt}\,dt\right.
\end{aligned}$$

$$
\begin{aligned}
&+\left(e^{l_e(M_i-t_{2i})}-1\right)\left\{\int_0^{t_{1i}} D_i\,dt+\int_{t_{1i}}^{t_{2i}} D_i\,dt\right\}\Bigg]\\
&= p_i\left[\int_0^{t_{1i}} f(p_i,W_i)t(e^{l_e t}-e^{-rt})\,dt+\int_{t_{1i}}^{t_{2i}} f(p_i,q_i(t))t(e^{l_e t}-e^{-rt})\,dt\right]\\
&\quad+\left(e^{l_e(M_i-t_{2i})}-1\right)p_i\left[\int_0^{t_{1i}} f(p_i,W_i)\,dt+\int_{t_{1i}}^{t_{2i}} f(p_i,q_i(t))\,dt\right]\\
&= p_i\left[\int_0^{t_{1i}} \frac{\alpha_i+\gamma W_i}{p_i^\beta}t(e^{l_e t}-e^{-rt})\,dt+\int_{t_{1i}}^{t_{2i}} \frac{\alpha_i+\gamma q_{2i}(t)}{p_i^\beta}t(e^{l_e t}-e^{-rt})\,dt\right]\\
&\quad+\left(e^{l_e(M_i-t_{2i})}-1\right)p_i\left[\int_0^{t_{1i}} \frac{\alpha_i+\gamma W_i}{p_i^\beta}\,dt+\int_{t_{1i}}^{t_{2i}} \frac{\alpha_i+\gamma q_{2i}(t)}{p_i^\beta}\,dt\right]\\
&= p_i\left[\frac{\alpha_i+\gamma W_i}{p_i^\beta}\left\{\left(\frac{t_{1i}e^{l_e t_{1i}}}{l_e}-\frac{e^{l_e t_{1i}}}{l_e^2}\right)+\frac{1}{l_e^2}+\left(\frac{t_{1i}e^{-rt_{1i}}}{r}+\frac{e^{-rt_{1i}}}{r^2}\right)-\frac{1}{r^2}\right\}\right.\\
&\quad+\frac{\alpha_i}{p_i^\beta}\left\{\left(\frac{t_{2i}e^{l_e t_{2i}}}{l_e}-\frac{e^{l_e t_{2i}}}{l_e^2}\right)-\left(\frac{t_{1i}e^{l_e t_{1i}}}{l_e}-\frac{e^{l_e t_{1i}}}{l_e^2}\right)+\left(\frac{t_{2i}e^{-rt_{2i}}}{r}+\frac{e^{-rt_{2i}}}{r^2}\right)\right.\\
&\quad\left.-\left(\frac{t_{1i}e^{-rt_{1i}}}{r}+\frac{e^{-rt_{1i}}}{r^2}\right)\right\}+\frac{\gamma K_{2i}}{p_i^\beta K_{1i}}\left\{\frac{t_{2i}e^{l_e t_{2i}}-t_{1i}e^{K_{1i}t_{2i}+(l_e-K_{1i})t_{1i}}}{l_e-K_{1i}}\right.\\
&\quad-\frac{e^{l_e t_{2i}}-e^{K_{1i}t_{2i}+(l_e-K_{1i})t_{1i}}}{(l_e-K_{1i})^2}-\frac{t_{2i}e^{l_e t_{2i}}-t_{1i}e^{l_e t_{1i}}}{l_e}+\frac{e^{l_e t_{2i}}-e^{l_e t_{1i}}}{l_e^2}\\
&\quad+\frac{t_{2i}e^{-rt_{2i}}-t_{1i}e^{K_{1i}t_{2i}-(K_{1i}+r)t_{1i}}}{K_{1i}+r}+\frac{e^{-rt_{2i}}-e^{K_{1i}t_{2i}-(K_{1i}+r)t_{1i}}}{(K_{1i}+r)^2}\\
&\quad\left.\left.-\frac{t_{2i}e^{-rt_{2i}}-t_{1i}e^{-rt_{1i}}}{r}-\frac{e^{-rt_{2i}}-e^{-rt_{1i}}}{r^2}\right\}\right]+\left(e^{l_e(M_i-t_{2i})}-1\right)\left[K_{4i}t_{1i}\right.\\
&\quad\left.+p_iK_{2i}(t_{2i}-t_{1i})+\frac{K_{2i}p_i\gamma}{K_{1i}^2p_i^\beta}\left\{\left(e^{K_{1i}(t_{2i}-t_{1i})}-1\right)-K_{1i}(t_{2i}-t_{1i})\right\}\right]. \qquad (30)
\end{aligned}
$$

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