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Article *in* Journal of Intelligent and Fuzzy Systems · April 2019

DOI: 10.3233/JIFS-18379

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Journal of Intelligent & Fuzzy Systems 36 (2019) 3725–3738
DOI:10.3233/JIFS-18379
IOS Press

Fully fuzzy inventory model with price-dependent demand and time varying holding cost under fuzzy decision variables

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Abstract. In this paper, we have considered an EOQ inventory model with a price-dependent demand and time varying holding cost in fuzzy environments by employing trapezoidal fuzzy numbers. A fully-fuzzy inventory model is developed where the input parameters and decision variables are fuzzified. For this fuzzy model, an expected value method of defuzzification is employed to find the estimate of the profit function in the fuzzy sense. In addition, a rigorous methodology is constructed to examined for the optimal solution of fully-fuzzy inventory model. The optimal policy for the developed model is determined using the proposed algorithm after defuzzification of the profit function. Finally, a numerical example is provided in order to determine the sensitiveness in the decision variables with respect to fuzziness in the components.

Keywords: Fully fuzzy inventory model, price-dependent demand, variable holding cost, fuzzy expected value, trapezoidal fuzzy variable

1. Introduction

The management of the inventory control system becomes more and more momentous for the enterprises in the real-life problems. Many world-wide researchers are fond in the solutions to the inventory management problem using various mathematical ethos. The first scientific approach to inventory management problem was the Harris-Wilson method popularly known as the economic order quantity (EOQ) formula. The EOQ formula gives the order quantity so as to meet customer service levels while minimizing the total inventory cost. This formula is generally recommended in problems where demand is constant.

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Many authors have considered several variations in the standard EOQ model. Recently, Chen [17] and San et al. [18], Feng et al. [35], Garai et al. [31], Guo and Liu [37] are developed on the EOQ model under imprecise demand and holding cost.

Many recent researchers are considered various type economic order quantity (EOQ) inventory models with variable demand and variable holding cost. In inventory models with variable demand rates, the demand for the given item is occupied to vary as a function of either the price, the stock level, or both. Inventory model in which the demand rate depends on the stock level are very often. Min and Zhou [9] constructed an inventory model for deteriorating items with a stock level dependent demand, partial backlogging, and a limit on the maximum inventory level. An EOQ inventory model with partial backlogging, stock-dependent demand and a control label

ISSN 1064-1246/19/$35.00

deterioration rate developed by Lee and Dye [8]. Zhengping [11] smeared an inventory model with price dependent demand, given partial demand information, in a supply chain with one retailer and one supplier. Moreover, many inventory models assume the unit holding cost to be variable. Ferguson et al. [4] developed an inventory model in which the holding cost has non-linear dependence on the storage time. Ghasemi and Afshar [6] considered two EOQ inventory model with variable holding costs, one with and the other without back order. Recently, many researchers considered inventory model in which both the holding cost and the demand rate are variable, the demand rate is assumed stock-dependent (cf. Alfares [3], Zhao and Zhong [10]). A partially integrated production-inventory model with interval valued inventory costs and variable demand is devloped by Bhunia et al. [33]. Mishra et al. [34] proposed an inventory model under price and stock dependent demand for controllable deterioration rate with shortages. Several inventory models considered demand reliance on other factors such as the product selling price and quality (cf. Wang et al. [38], Baten et al. [40], Chung and Cardenas-Barron [43], Cardenas-Barron et al. [42]). Kumar et al. [7] analyzed an EOQ inventory model under the assumption of price-dependent demand, where the holding cost is a time function of the trade credit for deteriorating items.

Fuzzy set theory, introduced by Zadeh [5], has been receiving considerable attention amidst researchers in production and inventory management, as well other areas. Mandal and Maiti [12] considered a nonlinear fuzzy modeling for a multi-item EOQ model with imprecise storage space and number of production run constraints where few input parameters were fuzzified. A multi-item fuzzy inventory model with inventory level dependent demand using possibility mean is developed by Garai et al. [36]. Mahata and Mahata [16] formulated an EOQ inventory model for deteriorating items under retailer partial trade credit financing in the supply chain. A fuzzy-rough inventory model with both stock-dependent demand and holding cost rate developed by Garai et al. [39]. Recently, Vijayan and Kumaran [13], Kazemi et al. [14], Mondal et al. [26], Agra et al. [28], Kundu et al. [29], Rodriguez et al. [27], Garai et al. [30], Mahata and Goswami [15] are investigated the economic order quantity model with fuzzy coefficients.

The literature review elicits that there is no EOQ model that has both its input parameters and decision variables fuzzified which are a limitation that this paper address. This paper chose a fuzzy EOQ model that has three decision variable; namely, order size, selling price and cycle time. It is unlike the work of Bjork [25] who fuzzified a single input parameter (demand) and a single decision variable (maximum inventory level). Similar problems to that of Bjork [25] and to the one in the paper are found in Chen and Wang [22], Khalil and Hassan [41], Vijayan and Kumaran [21], Garai et al. [32], Chen and chang [24] and Chen et al. [23].

In this paper, we have investigated the inventory problem with price-dependent demand and time varying holding cost by employing trapezoidal fuzzy numbers. The input parameters and decision variables are presented by trapezoidal fuzzy numbers in this model. For fully fuzzy inventory model, a method defuzzification, namely the expected value of the fuzzy variable, is employed to find the estimate of total profit in the fuzzy sense, and then the corresponding optimal fuzzy order size, fuzzy selling price and fuzzy cycle time are derived to maximize the total profit.

In spite of the above mentioned developments, following additions can also made in the formulation and solutions of the fully fuzzy inventory model with price-dependent demand and time varying holding cost.

- Fully fuzzy inventory model with price-dependent demand and time varying holding cost is developed.
- A rigorous methodology to convert the fully fuzzy inventory model equivalent to deterministic model have been presented.
- Till now, none has formulated a inventory model with the input parameters and decision variables are fuzzy variables.
- A numerical example has been provided to validate the proposed model as well as proposed methodology.

The rest of the paper organized as follows: In Section 2 we present the Alfares and Ghaithan crisp inventory model. Section 3 provides basic preliminaries for the fuzzy variable. In Section 4, we developed a fully fuzzy inventory model with price-dependent demand and time-varying holding cost, and discuss concavity proof of the profit function. The solution procedure of the proposed model discuss in Section 5. Section 6 illustrates the proposed inventory model with numerical examples. Section 7 provides a sensitivity analysis and discussion. Finally, the conclusion and scope of the future work plan have been made in Section 8.

2. Brief review of alfares and ghaithan model

Recently, Alfares and Ghaithan [20] developed an inventory model for an item with price-dependent demand, time-varying holding cost, and quantity discounts. They considered the following conditions.

(i) In typical EOQ-based inventory model, the demand rate (D) is a decreasing linear function of the selling price and purchase cost (c) is a decreasing step function of the order size Q according to all-units quantity discounts.
(ii) The holding cost (H) has two components: a constant component (g), and a variable component (h) that increases linearly with the length of storage time. The unit holding cost is proportional to the unit purchase cost (c), i.e., $H(t) = (g + ht)c$.
(iii) The unit purchase cost is subject to an all-units quantity discount. The unit purchase cost (c) is a decreasing step function of the order size (Q), i.e., $C(Q) = c$ if $q_{i-1} < Q \le q_i$.
(iv) The demand rate D is a linear decreasing function of the unit selling price P, i.e., $D(P) = a - bP$, where the selling price (P) must be lying in the range: $c < P < \frac{a}{b}$.
(v) The items do not deteriorate while kept in storage. Progressively more expensive and advanced storage facilities are used for longer storage duration, guaranteeing the preservation of quality of the stored items.
(vi) Shortages are not allowed.

The rate of decrease in the inventory level $q(t)$ is equal to the demand rate. This relationship is revealed by the following differential equation:

$$\frac{dq(t)}{dt} = -(a - bP) \qquad (1)$$

This yields to $q(t) = (a - bP)(T - t)Q = (a - bP)T$ and $T = \frac{Q}{a-bP}$.

The profit function ($TP(Q, P)$) includes the sales revenue, ordering cost, purchasing cost and holding cost. The total profit per cycle $TP(Q, P)$ is established as

$$TP(Q, P) = Sales\ revenue - Ordering\ cost - Purchasing\ Cost - Holding\ Cost$$

$$= P(a - bP) - c(a - bP) - \frac{K(a - bP)}{Q} - \frac{gcQ}{2} - \frac{hcQ^2}{6(a - bP)} \qquad (2)$$

The total cost per cycle $TC(Q, P)$, can be expressed as the following sum ordering cost , purchasing cost and holding cost components.

$$TC(Q, P) = Ordering\ cost + Purchasing\ Cost + Holding\ Cost$$

$$= \frac{K(a - bP)}{Q} + c(a - bP) + \frac{gcQ}{2} + \frac{hcQ^2}{6(a - bP)} \qquad (3)$$

We mentioned earlier, the input parameters and decision variables are described as crisp values in the profit function where it is maximized without obscurity in the results. Although these models provide some common understanding of the behaviour of the inventory under different assumptions, they are not able to presenting the real life situations. So, employing these models as they are, generally, leads to preposterous verdicts. Hence, using fuzzy set theory to solve inventory problems, which produces more precise results. In this study, we shall present all input parameters (K, c, a, b, g and h) and the decision variables (Q and P) as fuzzy numbers.

3. Preliminaries

Fuzzy set theory has owed as a powerful tool to quantitatively represent and manipulate the imprecision that sometimes governs the decision-making process. Fuzzy sets or fuzzy numbers can be used to encounter the imprecision by setting the values of the input parameters to be functions of triangular or trapezoidal shapes [1]. Some basic definitions, taken from [2], that are related to the fuzzy set theory are briefly reviewed below for the interest of the reciter.

Definition 3.1. $\tilde{a}$ is a fuzzy set in X (universe set). It is characterized by a membership function $\mu_{\tilde{a}}(x)$, which is associated with each element x, where x is a real number in the interval [0, 1]. The function value $\mu_{\tilde{a}}(x)$ is termed as the grade of membership of x in a.

Definition 3.2. The fuzzy set $\tilde{a}$ of the universe set X is convex, if $\mu_{\tilde{a}}(\eta x_1 + (1-\eta)x_2) \geq min(\mu_{\tilde{a}}(x_1), \mu_{\tilde{a}}(x_2))$ for $x_1, x_2 \in X$ and for $\eta \in [0, 1]$.

Definition 3.3. The fuzzy set $\tilde{a}$ of the universe set X is called a normal fuzzy set when there exist $x_i \in X; \mu_{\tilde{a}}(x_i) = 1$.

Definition 3.4. Any convex normalized fuzzy subset $\tilde{a}$ on $\mathbb{R}$ (where $\mathbb{R}$ is the set of real numbers) with membership function $\mu_{\tilde{a}} : \mathbb{R} \longrightarrow [0, 1]$ is called a fuzzy number. The fuzzy number $\tilde{a}$ is said to be a trapezoidal fuzzy number if it is determined by the crisp numbers (a_1, a_2, a_3, a_4), where $a_1 < a_2 < a_3 < a_4$, with membership function of the form

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ 1, & a_2 \leq x \leq a_3 \\ \frac{a_4-x}{a_4-a_3}, & a_3 \leq x \leq a_4 \\ 0, & \text{otherwise} \end{cases}$$

when $a_2 = a_3$, the trapezoidal fuzzy number becomes a triangular fuzzy number.

3.1. Fuzzy arithmetic operations

Some fuzzy arithmetic operations under the functional principle [3] for trapezoidal fuzzy numbers are given below:

Let $\tilde{a} = (a_1, a_2, a_3, a_4)$ and $\tilde{b} = (b_1, b_2, b_3, b_4)$ be two trapezoidal fuzzy numbers. Then

(i) Addition
$\tilde{a} \oplus \tilde{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$

(ii) Subtraction
$\ominus\tilde{b} = (-b_4, -b_3, -b_2, -b_1)$, $\tilde{a} \ominus \tilde{b} = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1)$

(iii) Multiplication
If $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and b_4 all are positive real numbers, then $\tilde{a} \otimes \tilde{b} \approx (a_1b_1, a_2b_2, a_3b_3, a_4b_4)$

(iv) Division
If $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and b_4 are all positive real numbers, then $\tilde{a} \oslash \tilde{b} \approx \left(\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1}\right)$

(v) Scalar multiplication
Let $k \in \mathbb{R}$, then $k \otimes \tilde{a} = (ka_1, ka_2, ka_3, ka_4)$ for $k \geq 0$, $k \otimes \tilde{a} = (ka_4, ka_3, ka_2, ka_1)$ for $k < 0$

3.2. Expected value of fuzzy variable

Definition 3.5. (Xu and Zhou [19]) Let $\tilde{a}$ be fuzzy variable on the possibility space $(X, P(X), Pos)$. The expected value of $\tilde{a}$ is defined by

$$\mathbf{E}^{Me}(\tilde{a}) = \int_0^{+\infty} Me\{\tilde{a} \geq x\}dx - \int_{-\infty}^{0} Me\{\tilde{a} \leq x\} \tag{4}$$

Similarly, we can define the expected value based on the *Pos*, *Nec*, *Cr* measures, which are the special case of fuzzy measure Me.

Definition 3.6. (Xu and Zhou [19]) Let $\tilde{a} = (a_1, a_2, a_3, a_4)$ be a TFN. Then the measure of $\tilde{a}$ defined as

$$Me(\tilde{a}) = \lambda Pos(\tilde{a}) + (1-\lambda)Nec(\tilde{a})$$

where $\lambda (0 \leq \lambda \leq 1)$ is the optimistic-pessimistic parameter, we determine the combined nature of decision maker λ, as follows

If $\lambda = 1$, then $Me = Pos$; it means the decision maker is optimistic and maximum chance of $\tilde{a}$ holds. If $\lambda = 0$, then $Me = Nec$; it means the decision maker is pessimistic and minimal chance of $\tilde{a}$ holds.
If $\lambda = 0.5$, then $Me = Cr$; it means the decision maker takes compromise attitude of $\tilde{a}$ holds. where Cr is the credibility measure and defined by $Cr = \frac{Pos+Nec}{2}$.

Theorem 3.1. (Xu and Zhou [19]) *Let $\tilde{a} = (a_1, a_2, a_3, a_4)$ be a trapezoidal fuzzy variable. Then its expected value is*

$$\mathbf{E}^{Me}(\tilde{a}) = \begin{cases} \frac{\lambda}{2}(a_1 + a_2) + \frac{1-\lambda}{2}(a_3 + a_4), & \text{if } a_4 \leq 0 \\ \frac{\lambda}{2}(a_1 + a_2) + \frac{\lambda a_4^2 - (1-\lambda)a_3^2}{2(a_4 - a_3)}, & \text{if } a_3 \leq 0 \leq a_4 \\ \frac{\lambda}{2}(a_1 + a_2 + a_3 + a_4), & \text{if } a_2 \leq 0 \leq a_3 \\ \frac{(1-\lambda)a_2^2 - \lambda a_1^2}{2(a_2 - a_1)} + \frac{\lambda}{2}(a_3 + a_4), & \text{if } a_1 \leq 0 \leq a_2 \\ \frac{1-\lambda}{2}(a_1 + a_2) + \frac{\lambda}{2}(a_3 + a_4), & \text{if } 0 \leq a_1 \end{cases} \tag{5}$$

Remark 3.1. If $\lambda = 0.5$, then

$$\mathbf{E}^{Cr}(\tilde{a}) = \frac{a_1 + a_2 + a_3 + a_4}{4}$$

Definition 3.7. Let $\tilde{a} = (a_1, a_2, a_3, a_4)$ be a trapezoidal fuzzy number, then the fuzzy expected value of $\tilde{a}$ is defined by

$$\mathbf{E}(\tilde{a}) = \frac{a_1 + a_2 + a_3 + a_4}{4}$$

4. Fuzzy inventory model with price-dependent demand and time varying holding cost

We thereby discuss the fuzzy inventory model for items with quantity discount and without shortage. In this section, the model presented in Section 2 is fully fuzzified, i.e by fuzzifying the input parameters (P, Q, K, h, c, g, a and b) and the decision variables (P and Q). Here, we assume that each input parameter is a non-negative trapezoidal fuzzy number consisting of nine components as:

Selling price: $\tilde{P} = (P_1, P_2, P_3, P_4)$, order size: $\tilde{Q} = (Q_1, Q_2, Q_3, Q_4)$, ordering cost: $\tilde{K} = (K_1, K_2, K_3, K_4)$, cycle time: $\tilde{T} = (T_1, T_2, T_3, T_4)$, unit purchasing cost: $\tilde{c} = (c_1, c_2, c_3, c_4)$, time varying holding cost: $\tilde{h} = (h_1, h_2, h_3, h_4)$, constant demand rate coefficient: $\tilde{a} = (a_1, a_2, a_3, a_4)$, price-dependent demand rate coefficient: $\tilde{b} = (b_1, b_2, b_3, b_4)$, constant holding cost coefficient: $\tilde{g} = (g_1, g_2, g_3, g_4)$. The full-fuzzy form of the total profit in Equation (2) is given as

$$\begin{aligned}\tilde{TP}(\tilde{Q}, \tilde{P}) = &\tilde{P} \otimes (\tilde{a} \ominus \tilde{b} \otimes \tilde{P}) \ominus (\tilde{c} \otimes (\tilde{a} \ominus \tilde{b} \otimes \tilde{P}))\\ &\ominus ((\tilde{K} \otimes (\tilde{a} \ominus \tilde{b} \otimes \tilde{P})) \oslash \tilde{Q})\\ &\ominus ((\tilde{g} \otimes \tilde{c} \otimes \tilde{Q}) \oslash 2) \ominus ((\tilde{h} \otimes \tilde{c} \otimes \tilde{Q}^2)\\ &\oslash (6 \otimes (\tilde{a} \ominus \tilde{b} \otimes \tilde{P}))) \end{aligned} \tag{6}$$

Similarly, the fully fuzzy form of the total cost in Equation (3) is given by

$$\begin{aligned}\tilde{TC}(\tilde{Q}, \tilde{P}) = &((\tilde{K} \otimes (\tilde{a} \ominus \tilde{b} \otimes \tilde{P})) \oslash \tilde{Q})\\ &\oplus (\tilde{c} \otimes (\tilde{a} \ominus \tilde{b} \otimes \tilde{P}))\\ &\oplus ((\tilde{g} \otimes \tilde{c} \otimes \tilde{Q}) \oslash 2)\\ &\oplus ((\tilde{h} \otimes \tilde{c} \otimes \tilde{Q}^2)\\ &\oslash (6 \otimes (\tilde{a} \ominus \tilde{b} \otimes \tilde{P}))) \end{aligned} \tag{7}$$

where $\oplus, \ominus, \otimes, \oslash$ are the fuzzy arithmetical operations under the Extension Principle and

$$\begin{aligned}&\tilde{P} \otimes (\tilde{a} \ominus \tilde{b} \otimes \tilde{P})\\ &= (P_1, P_2, P_3, P_4) \otimes ((a_1, a_2, a_3, a_4)\\ &\quad \ominus (b_1, b_2, b_3, b_4) \otimes (P_1, P_2, P_3, P_4))\\ &= (a_1 P_1 - b_4 P_1^2, a_2 P_2 - b_3 P_2^2,\\ &\quad a_3 P_3 - b_2 P_3^2, a_4 P_4 - b_1 P_4^2) \end{aligned} \tag{8}$$

$$\begin{aligned}&\ominus(\tilde{c} \otimes (\tilde{a} \ominus \tilde{b} \otimes \tilde{P}))\\ &= \ominus(-c_4, -c_3, -c_2, -c_1)\\ &\quad \otimes(a_1 - b_4 P_1, a_2 - b_3 P_2,\\ &\quad a_3 - b_2 P_3, a_4 - b_1 P_4)\\ &= ((b_4 P_1 - a_1)c_4, (b_3 P_2 - a_2)c_3,\\ &\quad (b_2 P_3 - a_3)c_2, (b_1 P_4 - a_4)c_1) \end{aligned} \tag{9}$$

$$\begin{aligned}&\ominus((\tilde{K} \otimes (\tilde{a} \ominus \tilde{b} \otimes \tilde{P})) \oslash \tilde{Q})\\ &= ((-K_4, -K_3, -K_2, -K_1)\\ &\quad \otimes(a_1 - b_4 P_1, a_2 - b_3 P_2,\\ &\quad a_3 - b_2 P_3, a_4 - b_1 P_4))\\ &\quad \oslash(Q_1, Q_2, Q_3, Q_4)\\ &= \left(\frac{(b_4 P_1 - a_1)K_4}{Q_4}, \frac{(b_3 P_2 - a_2)K_3}{Q_3},\right.\\ &\quad \left.\frac{(b_2 P_3 - a_3)K_2}{Q_2}, \frac{(b_1 P_4 - a_4)K_1}{Q_1}\right) \end{aligned} \tag{10}$$

$$\begin{aligned}\ominus((\tilde{g} \otimes \tilde{c} \otimes \tilde{Q}) \oslash 2) = &((-g_4, -g_3, -g_2, -g_1)\\ &\otimes(c_1, c_2, c_3, c_4)\\ &\otimes(Q_1, Q_2, Q_3, Q_4)) \oslash 2\\ &= \left(-\frac{g_4 c_1 Q_1}{2}, -\frac{g_3 c_2 Q_2}{2},\right.\\ &\quad \left.-\frac{g_2 c_3 Q_3}{2}, -\frac{g_1 c_4 Q_4}{2}\right) \end{aligned} \tag{11}$$

$$\begin{aligned}&\ominus((\tilde{h} \otimes \tilde{c} \otimes \tilde{Q}^2) \oslash (6 \otimes (\tilde{a} \ominus \tilde{b} \otimes \tilde{P})))\\ &= \ominus((h_1, h_2, h_3, h_4) \otimes (c_1, c_2, c_3, c_4)\\ &\quad \otimes(Q_1^2, Q_2^2, Q_3^2, Q_4^2))\\ &\quad \oslash(6 \otimes (a_1 - b_4 P_1, a_2 - b_3 P_2,\\ &\quad a_3 - b_2 P_3, a_4 - b_1 P_4)) \end{aligned}$$

$$= \left(-\frac{h_4 c_1 Q_1^2}{6(a_4 - b_1 P_4)}, -\frac{h_3 c_2 Q_2^2}{6(a_3 - b_2 P_3)}, -\frac{h_2 c_3 Q_3^2}{6(a_2 - b_3 P_2)}, -\frac{h_1 c_4 Q_4^2}{6(a_1 - b_4 P_1)} \right) \quad (12)$$

$$\tilde{T} = (T_1, T_2, T_3, T_4) = \left(\frac{Q_1}{a_4 - b_1 P_4}, \frac{Q_2}{a_3 - b_2 P_3}, \frac{Q_3}{a_2 - b_3 P_2}, \frac{Q_4}{a_1 - b_4 P_1} \right)$$

Substituting Equation (8)-(12) in Equation (6), the fuzzy total profit function is represented as

$$\tilde{TP}(\tilde{Q}, \tilde{P}) = (Tp_1, Tp_2, Tp_3, Tp_4) \quad (13)$$

Where,

$$Tp_1 = (a_1 P_1 - b_4 P_1^2) + (b_4 P_1 - a_1) c_4 + \frac{(b_4 P_1 - a_1) K_4}{Q_4} - \frac{g_4 c_1 Q_1}{2} - \frac{h_4 c_1 Q_1^2}{6(a_4 - b_1 P_4)}$$

$$Tp_2 = (a_2 P_2 - b_3 P_2^2) + (b_3 P_2 - a_2) c_3 + \frac{(b_3 P_2 - a_2) K_3}{Q_3} - \frac{g_3 c_2 Q_2}{2} - \frac{h_3 c_2 Q_2^2}{6(a_3 - b_2 P_3)}$$

$$Tp_3 = (a_3 P_3 - b_2 P_3^2) + (b_2 P_3 - a_3) c_2 + \frac{(b_2 P_3 - a_3) K_2}{Q_2} - \frac{g_2 c_3 Q_3}{2} - \frac{h_2 c_3 Q_3^2}{6(a_2 - b_3 P_2)}$$

$$Tp_4 = (a_4 P_4 - b_1 P_4^2) + (b_1 P_4 - a_4) c_1 + \frac{(b_1 P_4 - a_4) K_1}{Q_1} - \frac{g_1 c_4 Q_4}{2} - \frac{h_1 c_4 Q_4^2}{6(a_1 - b_4 P_1)}$$

Further, using the equation (7) & (8)-(12), the fuzzy total cost function is given by

$$\tilde{TC}(\tilde{Q}, \tilde{P}) = (Tc_1, Tc_2, Tc_3, Tc_4) \quad (14)$$

Where,

$$Tc_1 = \frac{K_1(a_1 - b_4 P_1)}{Q_4} + c_1(a_1 - b_4 P_1) + \frac{g_1 c_1 Q_1}{2} + \frac{h_1 c_1 Q_1^2}{6(a_4 - b_1 P_4)}$$

$$Tc_2 = \frac{K_2(a_2 - b_3 P_2)}{Q_3} + c_2(a_2 - b_3 P_2) + \frac{g_2 c_2 Q_2}{2} + \frac{h_2 c_2 Q_2^2}{6(a_3 - b_2 P_3)}$$

$$Tc_3 = \frac{K_3(a_3 - b_2 P_3)}{Q_2} + c_3(a_3 - b_2 P_3) + \frac{g_3 c_3 Q_3}{2} + \frac{h_3 c_3 Q_3^2}{6(a_2 - b_3 P_2)}$$

$$Tc_4 = \frac{K_4(a_4 - b_1 P_4)}{Q_1} + c_4(a_4 - b_1 P_4) + \frac{g_4 c_4 Q_4}{2} + \frac{h_4 c_4 Q_4^2}{6(a_1 - b_4 P_1)}$$

We defuzzify $\tilde{TP}(\tilde{Q}, \tilde{P})$, $\tilde{TC}(\tilde{Q}, \tilde{P})$ by formula (5) and obtain the expected value representation of $\tilde{TP}(\tilde{Q}, \tilde{P})$, $\tilde{TC}(\tilde{Q}, \tilde{P})$ as follows:

$$\begin{aligned} E(\tilde{TP}(\tilde{Q}, \tilde{P})) &= \frac{1}{4}\Big[(a_1 P_1 - b_4 P_1^2) + (b_4 P_1 - a_1) c_4 + \frac{(b_4 P_1 - a_1) K_4}{Q_4} - \frac{g_4 c_1 Q_1}{2} - \frac{h_4 c_1 Q_1^2}{6(a_4 - b_1 P_4)} \\ &+ (a_2 P_2 - b_3 P_2^2) + (b_3 P_2 - a_2) c_3 + \frac{(b_3 P_2 - a_2) K_3}{Q_3} - \frac{g_3 c_2 Q_2}{2} - \frac{h_3 c_2 Q_2^2}{6(a_3 - b_2 P_3)} + (a_3 P_3 - b_2 P_3^2) \\ &+ (b_2 P_3 - a_3) c_2 + \frac{(b_2 P_3 - a_3) K_2}{Q_2} - \frac{g_2 c_3 Q_3}{2} - \frac{h_2 c_3 Q_3^2}{6(a_2 - b_3 P_2)} + (a_4 P_4 - b_1 P_4^2) \\ &+ (b_1 P_4 - a_4) c_1 + \frac{(b_1 P_4 - a_4) K_1}{Q_1} - \frac{g_1 c_4 Q_4}{2} - \frac{h_1 c_4 Q_4^2}{6(a_1 - b_4 P_1)}\Big] \end{aligned} \quad (15)$$

And

$$E(\tilde{TC}(\tilde{Q},\tilde{P})) = \frac{1}{4}\Bigg[\frac{K_1(a_1-b_4P_1)}{Q_4}+c_1(a_1-b_4P_1)+\frac{g_1c_1Q_1}{2} + \frac{h_1c_1Q_1^2}{6(a_4-b_1P_4)}+\frac{K_2(a_2-b_3P_2)}{Q_3}+c_2(a_2-b_3P_2) + \frac{g_2c_2Q_2}{2}+\frac{h_2c_2Q_2^2}{6(a_3-b_2P_3)}+\frac{K_3(a_3-b_2P_3)}{Q_2} + c_3(a_3-b_2P_3)+\frac{g_3c_3Q_3}{2}+\frac{h_3c_3Q_3^2}{6(a_2-b_3P_2)} + \frac{K_4(a_4-b_1P_4)}{Q_1}+c_4(a_4-b_1P_4) + \frac{g_4c_4Q_4}{2}+\frac{h_4c_4Q_4^2}{6(a_1-b_4P_1)}\Bigg] \tag{16}$$

with $0 < Q_1 \le Q_2 \le Q_3 \le Q_4$ and $0 < P_1 \le P_2 \le P_3 \le P_4$.

4.1. Concavity of the profit function

To check whether the profit function is concave, we determine its Hessian matrix

$$H(E(\tilde{Q},\tilde{P})) = \begin{pmatrix}
\frac{\partial^2E}{\partial Q_1^2} & \frac{\partial^2E}{\partial Q_1\partial Q_2} & \frac{\partial^2E}{\partial Q_1\partial Q_3} & \frac{\partial^2E}{\partial Q_1\partial Q_4} & \frac{\partial^2E}{\partial Q_1\partial P_1} & \frac{\partial^2E}{\partial Q_1\partial P_2} & \frac{\partial^2E}{\partial Q_1\partial P_3} & \frac{\partial^2E}{\partial Q_1\partial P_4} \\
\frac{\partial^2E}{\partial Q_2\partial Q_1} & \frac{\partial^2E}{\partial Q_2^2} & \frac{\partial^2E}{\partial Q_2\partial Q_3} & \frac{\partial^2E}{\partial Q_2\partial Q_4} & \frac{\partial^2E}{\partial Q_2\partial P_1} & \frac{\partial^2E}{\partial Q_2\partial P_2} & \frac{\partial^2E}{\partial Q_2\partial P_3} & \frac{\partial^2E}{\partial Q_2\partial P_4} \\
\frac{\partial^2E}{\partial Q_3\partial Q_1} & \frac{\partial^2E}{\partial Q_3\partial Q_2} & \frac{\partial^2E}{\partial Q_3^2} & \frac{\partial^2E}{\partial Q_3\partial Q_4} & \frac{\partial^2E}{\partial Q_3\partial P_1} & \frac{\partial^2E}{\partial Q_3\partial P_2} & \frac{\partial^2E}{\partial Q_3\partial P_3} & \frac{\partial^2E}{\partial Q_3\partial P_4} \\
\frac{\partial^2E}{\partial Q_4\partial Q_1} & \frac{\partial^2E}{\partial Q_4\partial Q_2} & \frac{\partial^2E}{\partial Q_4\partial Q_3} & \frac{\partial^2E}{\partial Q_4^2} & \frac{\partial^2E}{\partial Q_4\partial P_1} & \frac{\partial^2E}{\partial Q_4\partial P_2} & \frac{\partial^2E}{\partial Q_4\partial P_3} & \frac{\partial^2E}{\partial Q_4\partial P_4} \\
\frac{\partial^2E}{\partial P_1\partial Q_1} & \frac{\partial^2E}{\partial P_1\partial Q_2} & \frac{\partial^2E}{\partial P_1\partial Q_3} & \frac{\partial^2E}{\partial P_1\partial Q_4} & \frac{\partial^2E}{\partial P_1^2} & \frac{\partial^2E}{\partial P_1\partial P_2} & \frac{\partial^2E}{\partial P_1\partial P_3} & \frac{\partial^2E}{\partial P_1\partial P_4} \\
\frac{\partial^2E}{\partial P_2\partial Q_1} & \frac{\partial^2E}{\partial P_2\partial Q_2} & \frac{\partial^2E}{\partial P_2\partial Q_3} & \frac{\partial^2E}{\partial P_2\partial Q_4} & \frac{\partial^2E}{\partial P_2\partial P_1} & \frac{\partial^2E}{\partial P_2^2} & \frac{\partial^2E}{\partial P_2\partial P_3} & \frac{\partial^2E}{\partial P_2\partial P_4} \\
\frac{\partial^2E}{\partial P_3\partial Q_1} & \frac{\partial^2E}{\partial P_3\partial Q_2} & \frac{\partial^2E}{\partial P_3\partial Q_3} & \frac{\partial^2E}{\partial P_3\partial Q_4} & \frac{\partial^2E}{\partial P_3\partial P_1} & \frac{\partial^2E}{\partial P_3\partial P_2} & \frac{\partial^2E}{\partial P_3^2} & \frac{\partial^2E}{\partial P_3\partial P_4} \\
\frac{\partial^2E}{\partial P_4\partial Q_1} & \frac{\partial^2E}{\partial P_4\partial Q_2} & \frac{\partial^2E}{\partial P_4\partial Q_3} & \frac{\partial^2E}{\partial P_4\partial Q_4} & \frac{\partial^2E}{\partial P_4\partial P_1} & \frac{\partial^2E}{\partial P_4\partial P_2} & \frac{\partial^2E}{\partial P_4\partial P_3} & \frac{\partial^2E}{\partial P_4^2}
\end{pmatrix} \tag{17}$$

The leading principle minors of $H(E(\tilde{Q},\tilde{P}))$ are $|D_{11}| < 0, |D_{22}| > 0, |D_{33}| < 0, |D_{44}| > 0$, $|D_{55}| < 0$, $|D_{66}| > 0$, $|D_{77}| < 0$ and $|D_{88}| > 0$. Appendix A shows that the values of the all leading principle minors are alternate sign for all value of $\tilde{Q} = (Q_1, Q_2, Q_3, Q_4)$ and $\tilde{P} = (P_1, P_2, P_3, P_4)$. Therefore, the Hessian matrix $H(E(\tilde{Q},\tilde{P}))$ is negative definite and consequently the profit function $\tilde{TP}(\tilde{Q},\tilde{P})$ is concave.

5. Solution procedure

In this section, the solution procedure for the fuzzy EOQ model is presented. Since our objective is to maximize the total profit; therefore the necessary conditions for maximizing the profit are given by

$$\frac{\partial E(\tilde{Q},\tilde{P})}{\partial Q_1}=0,\ \frac{\partial E(\tilde{Q},\tilde{P})}{\partial Q_2}=0,\ \frac{\partial E(\tilde{Q},\tilde{P})}{\partial Q_3}=0,\ \frac{\partial E(\tilde{Q},\tilde{P})}{\partial Q_4}=0,\ \frac{\partial E(\tilde{Q},\tilde{P})}{\partial P_1}=0,\ \frac{\partial E(\tilde{Q},\tilde{P})}{\partial P_2}=0,\ \frac{\partial E(\tilde{Q},\tilde{P})}{\partial P_3}=0,\ \frac{\partial E(\tilde{Q},\tilde{P})}{\partial P_4}=0 \tag{18}$$

Which gives the optimal values of $Q_1, Q_2, Q_3, Q_4, P_1, P_2, P_3$ and P_4. The sufficient conditions for maximizing the profit function $\tilde{TP}(\tilde{Q},\tilde{P})$ are $|D_{11}| < 0, |D_{22}| > 0, |D_{33}| < 0, |D_{44}| > 0, |D_{55}| < 0$, $|D_{66}| > 0$, $|D_{77}| < 0$ and $|D_{88}| > 0$, where $D_{11}, D_{22}, D_{33}, D_{44}, D_{55}, D_{66}, D_{77}$ and D_{88} are the leading principle minors of the Hessian matrix $H(E(\tilde{Q},\tilde{P}))$.

The optimal solution of proposed fuzzy inventory model can be obtained by using the following algorithm:

Algorithm:

(i) In put all the fuzzy parameters of the inventory model
(ii) For solving the fuzzy optimization problem for fuzzy EOQ inventory model, do the following:

Step-1: The objective function $E(\tilde{Q}, \tilde{P})$ of the optimization problem contained eight unknown variables(Q_1, Q_2, Q_3, Q_4, P_1, P_2, P_3 and P_4).
Step-2: Find the partial derivative of $E(\tilde{Q}, \tilde{P})$ with respect to each variables Q_1, Q_2, Q_3, Q_4, P_1, P_2, P_3 and P_4.
Step-3: Set each of the partial derivative equal to zero to get

$$\frac{\partial E(\tilde{Q}, \tilde{P})}{\partial Q_1} = 0, \frac{\partial E(\tilde{Q}, \tilde{P})}{\partial Q_2} = 0,$$
$$\frac{\partial E(\tilde{Q}, \tilde{P})}{\partial Q_3} = 0, \frac{\partial E(\tilde{Q}, \tilde{P})}{\partial Q_4} = 0,$$
$$\frac{\partial E(\tilde{Q}, \tilde{P})}{\partial P_1} = 0, \frac{\partial E(\tilde{Q}, \tilde{P})}{\partial P_2} = 0,$$
$$\frac{\partial E(\tilde{Q}, \tilde{P})}{\partial P_3} = 0, \frac{\partial E(\tilde{Q}, \tilde{P})}{\partial P_4} = 0 \qquad (19)$$

Step-4: Determine the value of Q_1, Q_2, Q_3, Q_4, P_1, P_2, P_3 and P_4 (using Lingo-14.0). If the total profit function $TP(\tilde{Q}, \tilde{P})$ attained maximum value for Q_1, Q_2, Q_3, Q_4, P_1, P_2, P_3 and P_4, then the values of Q_1, Q_2, Q_3, Q_4, P_1, P_2, P_3 and P_4 are called optimal solution for this inventory problem.

6. Numerical example

In this section, numerical examples are presented to illustrate the behaviour of the proposed model. We have developed in Section-3 of the crisp model [20]. This results compared with the crisp case consider the parameters of Alfares and Ghaithan [20].

Consider an inventory situation with crisp parameters having the following values (Alfares and Ghaithan [20]): ordering cost $K = 520$, unit purchasing cost $c = 4.75$, constant holding cost coefficient $g = 0.2$, time-varying holding cost coefficient $h = 0.05$, constant demand rate coefficient $a = 100$, price-dependent demand rate coefficient $b = 1.5$.

The optimal order size Q, optimal selling price per unit P, the optimal cycle time T, the maximum total profit $TP(Q, P)$ and the minimum total cost $TC(Q, P)$ of crisp case can be derived easily from (1), (2) and (3) respectively. We obtain $Q = 200$ units, $P = 36.52$ units, $T = 4.423$, $TC(Q, P) = 444.23$ units and $TP(Q, P) = 1207.20$ units.

We set some trapezoidal fuzzy numbers $\tilde{K} = (500, 515, 525, 540)$, $\tilde{c} = (3.95, 4.10, 4.35, 5.05)$, $\tilde{a} = (87, 91, 99, 117)$, $\tilde{b} = (1.2, 1.5, 1.8, 2.0)$, $\tilde{g} = (0.10, 0.15, 0.20, 0.35)$ and $\tilde{h} = (0.02, 0.05, 0.06, 0.07)$ of the input parameters K, c, a, b, g and h only to represent the components of fuzzy model developed in Section-3. For each of these parameters, the variations in the values are arranged arbitrary and their defuzzified values are determined by applying the expected value mean method. Using the proposed solution procedure, our optimal result is as follows: $\tilde{Q} = (167.42, 177.06, 178.43, 209.92)$ units, $\tilde{P} = (25.28, 28.55, 36.18, 52.00)$ units, $\tilde{T} = (3.066, 3.958, 4.504, 5.837)$, $TC(Q, P) = 442.99$ units and $TP(Q, P) = 1212.34$ units

7. Sensitivity analysis and discussion

In order to assess the relative influence of different input parameters on the solution attribute, a systematic sensitivity analysis was performed on the above example. The pic value of each given fuzzy parameters ($\tilde{a}, \tilde{c}, \tilde{K}$ and $\tilde{b}$) was changed, one at time, in relative steps of 20 % (-20%, -40%, +20 % and +40%) and the effect on the optimum solutions was noted. Since four new values were considered for each of the four parameters, sensitivity analysis required the solution of 16 spare problems. Table 1, Table 2, Table 3, and Table 4 are shows the impact of changes in the input parameters on the decision variables: $\tilde{P}$, $\tilde{Q}$, $\tilde{T}$ and objective function TP, TC. From Table 1–4, the following discussions are made:

(i) From Table 1–4, it is observed that the total profit (TP) increases with higher value of $\tilde{a}$, lower values of $\tilde{c}, \tilde{b}$ and $\tilde{K}$ and decreases with lower values of $\tilde{a}$, higher values of $\tilde{c}, \tilde{b}$ and $\tilde{K}$ (cf. Figs. 1 & 2).
(ii) From Table 1–4, it is clear that the total cost (TC) decreases with higher value of $\tilde{b}$, lower values of $\tilde{c}, \tilde{a}$ and $\tilde{K}$ and increases with lower value of $\tilde{b}$, higher values of $\tilde{c}, \tilde{a}$ and $\tilde{K}$ (cf. Figs. 3 & 4).
(iii) From Table 1–3, it is clearly visible that the order size $\tilde{Q}$ increases with lower values of $\tilde{c}, \tilde{b}$, the higher values of $\tilde{a}$ and decreases with higher values of $\tilde{c}, \tilde{b}$, the lower values of $\tilde{a}$.
(iv) From Table 1–2, it is observed that the selling price $\tilde{P}$ increases with higher values of $\tilde{a}, \tilde{c}$ and decreases with lower values of $\tilde{a}, \tilde{c}$.

Table 1
Impact of $\tilde{a}$ on the optimal replenishment policy

Parameters	Original value	New value	$\tilde{Q}$	$\tilde{P}$	$\tilde{T}$	TP	TC
$\tilde{a}$	(87.00, 91.00, 99.00, 117)	(87.00, 91.00, 99.00, 117)	(167.42, 177.06, 178.71, 209.92)	(25.28, 28.60, 36.24, 52.00)	(3.066, 3.958, 4.504, 5.837)	1212.33	443.51
	-20 %	(87.00, 90.80, 98.80, 117)	(167.42, 176.83, 178.16, 209.92)	(25.28, 28.50, 36.11, 52.00)	(3.066, 3.984, 4.510, 5.760)	1209.74	443.14
	-40 %	(87.00, 90.60, 98.60, 117)	(167.42, 176.59, 177.88, 209.92)	(25.28, 28.44, 36.05, 52.00)	(3.066, 3.966, 4.513, 5.760)	1207.16	442.85
	+20 %	(87.00, 91.20, 99.20, 117)	(167.42, 177.30, 178.71, 209.92)	(25.28, 28.60, 36.24, 52.00)	(3.066, 3.954, 4.499, 5.837)	1214.92	443.55
	+40 %	(87.00, 91.40, 99.40, 117)	(167.42, 177.54, 178.99, 209.92)	(25.28, 28.66, 36.31, 52.00)	(3.066, 3.951, 4.495, 5.837)	1217.52	443.65

Table 2
Impact of $\tilde{c}$ on the optimal replenishment policy

Parameters	Original value	New value	$\tilde{Q}$	$\tilde{P}$	$\tilde{T}$	TP	TC
$\tilde{c}$	(3.95, 4.10, 4.35, 5.05)	(3.95, 4.10, 4.35, 5.05)	(167.42, 177.06, 178.43, 209.92)	(25.28, 28.55, 36.18, 52.00)	(3.066, 3.958, 4.504, 5.837)	1212.33	442.99
	-20 %	(3.95, 3.90, 4.15, 5.05)	(167.42, 181.19, 182.45, 209.92)	(25.28, 28.42, 36.05, 52.00)	(3.066, 4.033, 4.579, 5.837)	1218.83	437.69
	-40 %	(3.95, 3.70, 3.95, 5.05)	(167.42, 185.60, 16.73, 209.92)	(25.28, 28.30, 35.92, 52.00)	(3.066, 4.113, 4.661, 5.760)	1225.41	431.72
	+20 %	(3.95, 4.30, 4.55, 5.05)	(167.42, 173.20, 174.64, 209.92)	(25.28, 28.68, 36.31, 52.00)	(3.066, 3.889, 4.435, 5.837)	1205.91	448.70
	+40 %	(3.95, 4.50, 4.75, 5.05)	(167.42, 169.56, 171.06, 209.92)	(25.28, 28.80, 36.43, 52.00)	(3.066, 3.822, 4.368, 5.837)	1199.58	454.37

Table 3
Impact of $\tilde{b}$ on the optimal replenishment policy

Parameters	Original value	New value	$\tilde{Q}$	$\tilde{P}$	$\tilde{T}$	TP	TC
$\tilde{b}$	(1.20, 1.50, 1.80, 2.00)	(1.20, 1.50, 1.80, 2.00)	(167.42, 177.06, 178.43, 209.92)	(25.28, 28.55, 36.18, 52.00)	(3.066, 3.958, 4.504, 5.837)	1212.33	442.99
	-20 %	(1.20, 1.30, 1.60, 2.00)	(167.42, 178.57, 180.24, 209.92)	(25.28, 31.70, 41.25, 52.00)	(3.066, 3.935, 4.474, 5.837)	1310.06	445.64
	-40 %	(1.20, 1.10, 1.40, 2.00)	(34.946, 176.58, 184.52, 209.92)	(25.28, 40.88, 57.86, 52.00)	(0.734, 4.473, 5.004, 5.837)	1508.08	575.00
	+20 %	(1.40, 1.70, 2.00, 2.00)	(167.42, 175.54, 176.60, 209.92)	(25.28, 26.03, 32.30, 52.00)	(3.066, 3.978, 4.535, 5.837)	1136.58	441.39
	+40 %	(1.40, 1.90, 2.00, 2.00)	(167.42, 173.99, 174.74, 209.92)	(23.98, 23.98, 29.95, 52.00)	(3.066, 4.006, 4.569, 5.837)	1075.17	439.83

(v) From the result shown in Table 1 & 2, it is observed that the cycle time $\tilde{T}$ increases with lower values of $\tilde{a}, \tilde{c}$ and decreases with higher values of $\tilde{a}, \tilde{c}$.

Table 4
Impact of $\tilde{K}$ on the optimal replenishment policy

Parameters	Original value	New value	$\tilde{Q}$	$\tilde{P}$	$\tilde{T}$	TP	TC
$\tilde{K}$	(500.00, 515.00, 525.00, 540.00)	(500.00, 515.00, 525.00, 540.00)	(167.42, 177.06, 178.43, 209.92)	(25.28, 28.55, 36.18, 52.00)	(3.066, 3.958, 4.504, 5.837)	1212.33	443.51
	-20 %	(500.00, 514.80, 524.80, 540.00)	(167.42, 177.04, 178.41, 209.92)	(25.28, 28.55, 36.18, 52.00)	(3.066, 3.957, 4.504, 5.837)	1212.35	443.49
	-40 %	(500.00, 514.60, 524.60, 540.00)	(167.42, 177.01, 178.38, 209.92)	(25.28, 28.55, 36.18, 52.00)	(3.066, 3.597, 4.503, 5.837)	1212.38	443.46
	+20 %	(500.00, 515.20, 525.20, 540.00)	(167.42, 177.09, 178.46, 209.92)	(25.28, 28.55, 36.18, 52.00)	(3.066, 3.959, 4.505, 5.837)	1212.31	443.53
	+40 %	(500.00, 515.40, 525.40, 540.00)	(167.42, 177.12, 178.49, 209.92)	(25.28, 28.55, 36.18, 52.00)	(3.066, 3.959, 4.506, 5.837)	1212.28	443.56

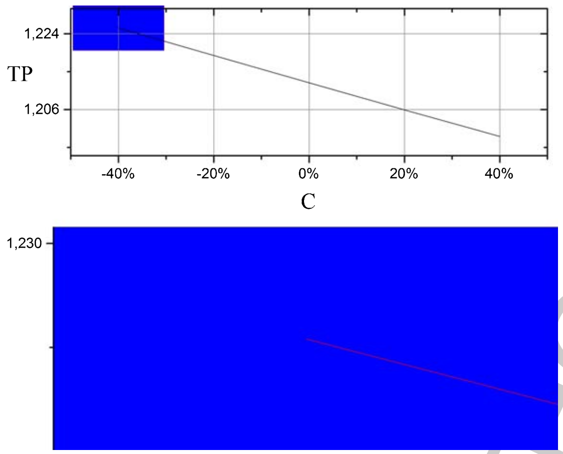

Fig. 1. $\tilde{c}$ Vs. Max TP.

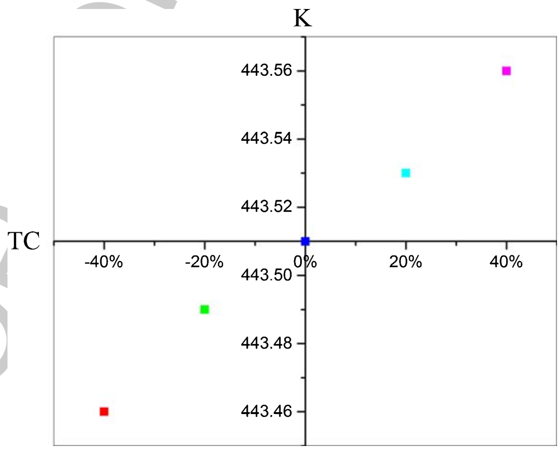

Fig. 3. $\tilde{K}$ Vs. Min TC.

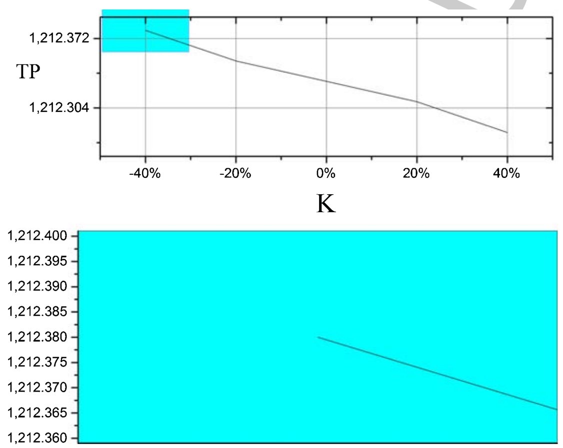

Fig. 2. $\tilde{K}$ Vs. Max TP.

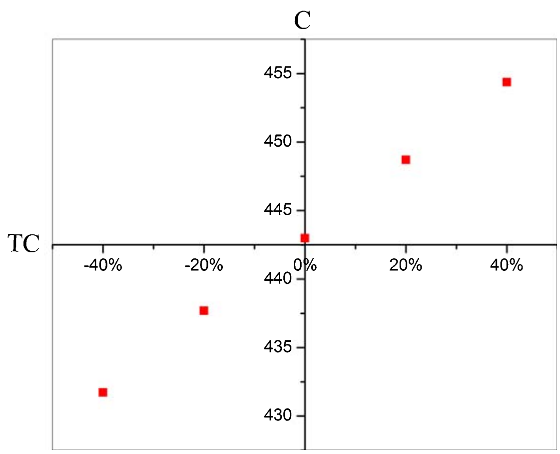

Fig. 4. $\tilde{c}$ Vs. Min TC.

Clearly, unit purchasing cost ($\tilde{c}$) and ordering cost ($\tilde{K}$) are the most capable factors on the profit function (TP) and the values of the variable $\tilde{Q}$, $\tilde{P}$, $\tilde{T}$ and total cost function (TC). This means that, in order to maximize profitability, companies should be more worried with increasing demand than with diminution costs.

8. Conclusion and future research directions

This paper presented a fully fuzzy inventory model with a variable demand, a variable holding cost, and a variable purchase cost. In this model, the input parameters are presented with fuzzy numbers, while the decision variables are treated as fuzzy numbers. The fully fuzzy inventory model was solved for trapezoidal fuzzy numbers using the Lagrangian optimization method. Numerical examples are carried out to investigate the behaviour of our proposed fuzzy model. We notice that the optimal solution of the proposed fuzzy inventory problem is more helpful from Alfares and Ghaithan [20] crisp model. The result of sensitivity analysis shows that decision variable and the total profit function affected by the two cost parameters, one purchasing cost ($\tilde{c}$) and another ordering cost ($\tilde{K}$). This means that, if the purchasing cost and ordering cost are reduced, then companies should reduce the unit selling price in order to boost the demand and increase their revenues.

The proposed fuzzy inventory model can be expended further many ways. The present fuzzy inventory model can be formulated with trapezoidal type demand or demand rate as a non-linear function of the selling price. Other possibilities include the consideration of shortages, time value of money and deteriorating items, etc.. Moreover, the present investigation can be extended to include imprecise environment such as fuzzy rough, fuzzy random, bi-fuzzy, etc.. This may enhance the trend value also.

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Appendix A. Proof of concavity of (6)

From (17), the first leading principle minor of $H(E(\tilde{Q}, \tilde{P}))$ is

$$D_{11} = -\frac{1}{4}\left(\frac{h_4c_1}{3(a_4 - b_1P_4)} - \frac{2(b_1P_4 - a_4)K_1}{Q_1^3}\right)$$
$$= -\frac{1}{4}\left(\frac{h_4c_1 + 6(a_4 - b_1P_4)^2K_1}{3(a_4 - b_1P_4)Q_1^3}\right) < 0 \quad (20)$$

The second principle minor of $H(E(\tilde{Q}, \tilde{P}))$ is given by

$$D_{22} = \frac{1}{16}\left(\frac{h_4c_1}{3(a_4 - b_1P_4)} - \frac{2(b_1P_4 - a_4)K_1}{Q_1^3}\right)\left(\frac{h_3c_2}{3(a_3 - b_2P_3)} - \frac{2(b_2P_3 - a_3)K_2}{Q_2^3}\right)$$
$$= \frac{1}{16}\left(\frac{h_4c_1 + 6(a_4 - b_1P_4)^2K_1}{3(a_4 - b_1P_4)Q_1^3}\right)\left(\frac{h_3c_2 + 6(a_3 - b_2P_3)^2K_2}{3(a_3 - b_2P_3)Q_2^3}\right) > 0 \quad (21)$$

The third principle minor of $H(E(\tilde{Q}, \tilde{P}))$ is

$$D_{33} = -\frac{1}{64}\left(\frac{h_4c_1}{3(a_4 - b_1P_4)} - \frac{2(b_1P_4 - a_4)K_1}{Q_1^3}\right)\left(\frac{h_3c_2}{3(a_3 - b_2P_3)} - \frac{2(b_2P_3 - a_3)K_2}{Q_2^3}\right)\left(\frac{h_2c_3}{3(a_2 - b_3P_2)} - \frac{2(b_3P_2 - a_2)K_3}{Q_3^3}\right)$$
$$= -\frac{1}{64}\left(\frac{h_4c_1 + 6(a_4 - b_1P_4)^2K_1}{3(a_4 - b_1P_4)Q_1^3}\right)\left(\frac{h_3c_2 + 6(a_3 - b_2P_3)^2K_2}{3(a_3 - b_2P_3)Q_2^3}\right)\left(\frac{h_2c_3 + 6(a_2 - b_3P_2)^2K_3}{3(a_2 - b_3P_2)Q_3^3}\right) < 0 \quad (22)$$

Similarly, we determined other leading principle minors $D_{44} > 0$, $D_{55} < 0$, $D_{66} > 0$, $D_{77} < 0$ and $D_{88} > 0$. All leading principle minors of $H(E(\tilde{Q}, \tilde{P}))$ are alternate sign. Consequently, the Hessian matrix $H(E(\tilde{Q}, \tilde{P}))$ is negative definite, and the profit function $\tilde{TP}(\tilde{Q}, \tilde{P})$ is concave.

$$\frac{\partial^2 E}{\partial Q_1^2} = -\frac{1}{4}\left(\frac{h_4c_1}{3(a_4 - b_1P_4)} - \frac{2(b_1P_4 - a_4)K_1}{Q_1^3}\right),$$
$$\frac{\partial^2 E}{\partial Q_1\partial Q_2} = 0,\ \frac{\partial^2 E}{\partial Q_1\partial Q_3} = 0,\ \frac{\partial^2 E}{\partial Q_1\partial Q_4} = 0,$$

$$\frac{\partial^2 E}{\partial Q_1\partial P_1} = 0,\ \frac{\partial^2 E}{\partial Q_1\partial P_2} = 0,\ \frac{\partial^2 E}{\partial Q_1\partial P_3} = 0,$$
$$\frac{\partial^2 E}{\partial Q_1\partial P_4} = -\frac{h_4b_1c_1Q_1}{12(a_4 - b_1P_4)} \quad (23)$$

$$\frac{\partial^2 E}{\partial Q_2^2} = -\frac{1}{4}\left(\frac{h_3c_2}{3(a_3 - b_2P_3)} - \frac{2(b_2P_3 - a_3)K_2}{Q_2^3}\right),$$
$$\frac{\partial^2 E}{\partial Q_2\partial Q_1} = 0,\ \frac{\partial^2 E}{\partial Q_2\partial Q_3} = 0,\ \frac{\partial^2 E}{\partial Q_2\partial Q_4} = 0,$$
$$\frac{\partial^2 E}{\partial Q_2\partial P_1} = 0,\ \frac{\partial^2 E}{\partial Q_2\partial P_2} = 0,$$
$$\frac{\partial^2 E}{\partial Q_2\partial P_3} = -\frac{h_3b_2c_2Q_2}{12(a_3 - b_2P_3)},\ \frac{\partial^2 E}{\partial Q_2\partial P_4} = 0 \quad (24)$$

$$\frac{\partial^2 E}{\partial Q_3^2} = -\frac{1}{4}\left(\frac{h_2c_3}{3(a_2 - b_3P_2)} - \frac{2(b_3P_2 - a_2)K_3}{Q_3^3}\right),$$
$$\frac{\partial^2 E}{\partial Q_3\partial Q_1} = 0,\ \frac{\partial^2 E}{\partial Q_3\partial Q_2} = 0,\ \frac{\partial^2 E}{\partial Q_3\partial Q_4} = 0,$$
$$\frac{\partial^2 E}{\partial Q_3\partial P_1} = 0,\ \frac{\partial^2 E}{\partial Q_3\partial P_2} = -\frac{h_2b_3c_3Q_3}{12(a_2 - b_3P_2)},$$
$$\frac{\partial^2 E}{\partial Q_3\partial P_3} = 0,\ \frac{\partial^2 E}{\partial Q_3\partial P_4} = 0 \quad (25)$$

$$\frac{\partial^2 E}{\partial Q_4^2} = -\frac{1}{4}\left(\frac{h_1c_4}{3(a_1 - b_4P_1)} - \frac{2(b_4P_1 - a_1)K_4}{Q_4^3}\right),$$
$$\frac{\partial^2 E}{\partial Q_4\partial Q_1} = 0,\ \frac{\partial^2 E}{\partial Q_4\partial Q_2} = 0,\ \frac{\partial^2 E}{\partial Q_4\partial Q_3} = 0,$$
$$\frac{\partial^2 E}{\partial Q_4\partial P_1} = -\frac{h_1b_4c_4Q_4}{12(a_1 - b_4P_1)},$$
$$\frac{\partial^2 E}{\partial Q_4\partial P_2} = 0,\ \frac{\partial^2 E}{\partial Q_4\partial P_3} = 0,\ \frac{\partial^2 E}{\partial Q_4\partial P_4} = 0 \quad (26)$$

$$\frac{\partial^2 E}{\partial P_1^2} = -\frac{1}{4}\left(2b_4 + \frac{h_1 c_4 b_4^2 Q_4^2}{3(a_1 - b_4 P_1)^3}\right),$$

$$\frac{\partial^2 E}{\partial P_1 \partial P_2} = 0, \frac{\partial^2 E}{\partial P_1 \partial P_3} = 0, \frac{\partial^2 E}{\partial P_1 \partial P_4} = 0,$$

$$\frac{\partial^2 E}{\partial P_1 \partial Q_1} = 0 \frac{\partial^2 E}{\partial P_1 \partial Q_2} = 0, \frac{\partial^2 E}{\partial P_1 \partial Q_3} = 0,$$

$$\frac{\partial^2 E}{\partial P_1 \partial Q_4} = -\left(\frac{h_1 c_4 Q_4 b_4}{3(a_1 - b_4 P_1)^2} + \frac{b_4}{Q_4^2}\right) \quad (27)$$

$$\frac{\partial^2 E}{\partial P_2^2} = -\frac{1}{4}\left(2b_3 + \frac{h_2 c_3 b_3^2 Q_3^2}{3(a_2 - b_3 P_2)^3}\right),$$

$$\frac{\partial^2 E}{\partial P_2 \partial P_1} = 0, \frac{\partial^2 E}{\partial P_2 \partial P_3} = 0, \frac{\partial^2 E}{\partial P_2 \partial P_4} = 0,$$

$$\frac{\partial^2 E}{\partial P_2 \partial Q_1} = 0, \frac{\partial^2 E}{\partial P_2 \partial Q_2} = 0,$$

$$\frac{\partial^2 E}{\partial P_2 \partial Q_3} = -\left(\frac{h_2 c_3 Q_3 b_3}{3(a_2 - b_3 P_2)^2} + \frac{b_3}{Q_3^2}\right),$$

$$\frac{\partial^2 E}{\partial P_2 \partial Q_4} = 0 \quad (28)$$

$$\frac{\partial^2 E}{\partial P_3^2} = -\frac{1}{4}\left(2b_2 + \frac{h_3 c_2 b_2^2 Q_2^2}{3(a_3 - b_2 P_3)^3}\right),$$

$$\frac{\partial^2 E}{\partial P_3 \partial P_1} = 0, \frac{\partial^2 E}{\partial P_3 \partial P_2} = 0, \frac{\partial^2 E}{\partial P_3 \partial P_4} = 0,$$

$$\frac{\partial^2 E}{\partial P_3 \partial Q_1} = 0$$

$$\frac{\partial^2 E}{\partial P_3 \partial Q_2} = -\left(\frac{h_3 c_2 Q_2 b_2}{3(a_3 - b_2 P_3)^2} + \frac{b_2}{Q_2^2}\right),$$

$$\frac{\partial^2 E}{\partial P_3 \partial Q_3} = 0, \frac{\partial^2 E}{\partial P_3 \partial Q_4} = 0 \quad (29)$$

$$\frac{\partial^2 E}{\partial P_4^2} = -\frac{1}{4}\left(2b_1 + \frac{h_4 c_1 b_1^2 Q_1^2}{3(a_4 - b_1 P_4)^3}\right),$$

$$\frac{\partial^2 E}{\partial P_4 \partial P_1} = 0, \frac{\partial^2 E}{\partial P_4 \partial P_2} = 0,$$

$$\frac{\partial^2 E}{\partial P_4 \partial P_3} = 0, \frac{\partial^2 E}{\partial P_4 \partial Q_2} = 0$$

$$\frac{\partial^2 E}{\partial P_4 \partial Q_1} = -\left(\frac{h_4 c_1 Q_1 b_1}{3(a_4 - b_1 P_4)^2} + \frac{b_1}{Q_1^2}\right),$$

$$\frac{\partial^2 E}{\partial P_4 \partial Q_3} = 0, \frac{\partial^2 E}{\partial P_4 \partial Q_4} = 0 \quad (30)$$