GRAPH THEORY AND ALGEBRAIC STRUCTURES
(MATH 2203)

Time Allotted : 3 hrs **Full Marks : 70**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 5 (five) from Group B to E, taking at least one from each group.

Candidates are required to give answer in their own words as far as practicable.

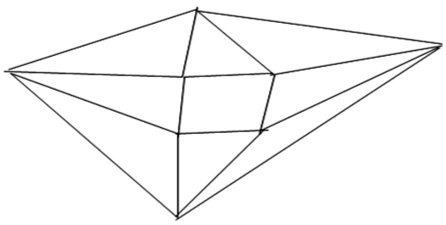
Group – A
(Multiple Choice Type Questions)

1. Choose the correct alternative for the following: **10 × 1 = 10**

(i) If the cyclic group G contains 8 distinct elements then the number of generator(s) of G is (are)
(a) 1 (b) 2 (c) 3 (d) 8

(ii) Let G be a simple connected planar graph with 13 vertices and 19 edges. Then, the number of faces in the planar embedding of the graph is
(a) 6 (b) 8 (c) 9 (d) 13

(iii) The chromatic number for the following graph is



(a) 2 (b) 3 (c) 4 (d) 5

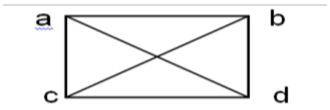
(iv) The order of the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ is
(a) 2 (b) 3 (c) 1 (d) 4

(v) Let $S = \{1,2,3,4, \dots, n\}$. Then S_n is
(a) an abelian group (b) a cyclic group
(c) contains even number of elements (d) a non-abelian group

(vi) Index of a subgroup H of a group G is 5 and its order is 3. The order of the group G is
(a) 8 (b) 10 (c) 15 (d) 25.

(vii) If x is an element of a group G and $O(x) = 5$, then
(a) $O(x^{10}) = 5$ (b) $O(x^{15}) = 5$ (c) $O(x^{23}) = 5$ (d) $O(x^{20}) = 5$

(viii)



G_1

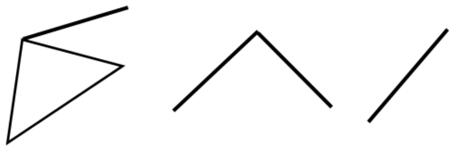
(a) G_1 is non-planar complete graph (b) G_1 is planar and complete graph
(c) G_1 is planar but not complete (d) G_1 is neither planar nor complete

(ix) Consider the binary relation $R = \{(x,y),(x,z),(z,x),(z,y)\}$ on the set $\{x,y,z\}$. Which one of the following is TRUE?
(a) R is symmetric but not antisymmetric
(b) R is not symmetric but antisymmetric
(c) R is both symmetric and antisymmetric
(d) R is neither symmetric nor antisymmetric.

(x) The unity element of the ring of matrices $\left\{\begin{pmatrix} x & y \\ -y & x \end{pmatrix} : x, y \in \mathbb{R}\right\}$ with respect to matrix addition and matrix multiplication is
(a) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Group – B

2. (a) Find the chromatic polynomial of the following disconnected graphG:



[(Evaluate/HOCQ)]

(b) State and prove Euler's formula for planar graph. [(Remember/LOCQ)]

6 + 6 = 12

3. (a) How many edges a planar graph must have with 5 regions and 7 vertices? Draw one such graph. [(Evaluate/HOCQ)]
(b) Prove that in any graph G, the constant term in its chromatic polynomial is zero. [(Analyze/IOCQ)]

6 + 6 = 12

Group – C

4. (a) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 6 \end{pmatrix}$

Compute each of the following: (i) α^{-1} (ii) $\beta\alpha$. [(Analyze/IOCQ)]

(b) Prove that in a group G for all a, b $\in$ G, the equation ax $=$ b has a unique solution in G. [(Analyze/IOCQ)]

6 + 6 = 12

5. (a) Prove that the order of an element of a group is same as that of its inverse. [(Apply/IOCQ)]

(b) Show that the set G of all ordered pairs *(a,b)* with $a \neq 0$, of real numbers a, b forms a group with operation '$\circ$' defined by, $(a,b) \circ (c,d)=(ac, bc+d)$
[(Understand/LOCQ)]

6 + 6 = 12

Group - D

6. (a) Prove that a finite group G is a cyclic group if and only if there exists an element $a \in G$ such that $O(a) = |G|$. [(Remember/LOCQ)]

(b) If a be an element of order n in a group G and p be prime to n, then prove that a^p is also of order n. [(Analyze/IOCQ)]

6 + 6 = 12

7. (a) State and prove Lagrange's theorem regarding the order of a subgroup of a finite group. [(Apply/IOCQ)]

(b) Let H be a subgroup of a group G and let $a, b \in G$. Prove that $aH = bH$ if and only if $a^{-1}b \in H$. [(Apply/LOCQ)]

6 + 6 = 12

Group – E

8. (a) Prove that a field is an integral domain. [(Remember/LOCQ)]

(b) Determine whether the given map $\phi: M_2(\mathbb{R}) \longrightarrow (\mathbb{R}, +)$ defined by $\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + b + c + d$ is a homomorphism, where $M_2(\mathbb{R})$ is the set of matrices of order 2.
[(Evaluate/HOCQ)]

6 + 6 = 12

9. (a) Let R be a ring. The centre of R is the subset Z(R) defined by

$$Z(R) = \{x \in R: xr = rx \; \forall \; r \in R\}.$$

Prove that Z(R) is a subring of R. [(Evaluate/HOCQ)]

(b) Examine if the ring of matrices $\left\{\begin{pmatrix} a & b \\ 2b & a \end{pmatrix}: a, b \in \mathbb{R}\right\}$ is a field.
[(Apply/LOCQ)]

6 + 6 = 12

Cognition Level	LOCQ	IOCQ	HOCQ
Percentage distribution	25	50	25

*LOCQ: Lower Order Cognitive Question; IOCQ: Intermediate Order Cognitive Question; HOCQ: Higher Order Cognitive Question