ADVANCED PROBABILITY AND INFORMATION THEORY (MATH 3222)

Time Allotted : 3 hrs. **Full Marks : 70**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 5 (five) from Group B to E, taking at least one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A
(Multiple Choice Type Questions)

1. Choose the correct alternative for the following: **10 × 1 = 10**

(i) Let X and Y be two random variables each taking three values $-1, 0, 1$ and having joint probability distribution

Y \ X	-1	0	1
-1	0.1	0	0.2
0	0.2	0.2	0.6
1	0	0.1	0.2

Then $E(Y|X=-1)$ is

(a) $\frac{1}{3}$ (b) 0 (c) 0.1 (d) $\frac{1}{2}$.

(ii) A coin is weighted so that its probability of landing heads is 25%. Suppose that the coin is flipped 40 times. Then a bound for the probability that it lands on head at least 35 times is

(a) $\frac{1}{4}$ (b) $\frac{3}{7}$ (c) $\frac{2}{7}$ (d) 1

(iii) Relative entropy $D(p||q)$ is
(a) concave in the pair (p, q)
(b) convex in the pair (p, q)
(c) neither convex nor concave in the pair (p, q)
(d) always negative.

(iv) If X and Y are two independent and identically distributed random variables, then $Pr(X = Y)$ is

(a) $\geq 2^{-H(X)}$ (b) $\leq 2^{-H(X)}$
(c) $= 2^{-I(X,Y)}$ (d) $= -H(X)$.

(v) The first four moments about the mean of distribution are $0, \mu_2, 0.7$ and 18.75. If the distribution is mesokurtic, the value of μ_2 is

(a) 2.5 (b) 3.5 (c) 0.5 (d) 1.5.

(vi) Following is the transition probability matrix of a Markov chain:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1-2a & 2a & 0 \\ a & 1-2a & a \\ 0 & 2a & 1-2a \end{bmatrix} \end{array}$$

then

(a) value of a is any real number (b) value of a is any positive real number
(c) value of a less than 0.5 (d) value of a is $[0, 0.5]$

(vii) If X and Y are independent random variables, then mutual information I(X, Y) of (X, Y) is

(a) 0 (b) $H(Y) + H(X/Y)$

(c) $H(Y)H(X)$ (d) 1

(viii) The probability that A speaks truth is $\frac{4}{5}$, while this probability for B is $\frac{3}{4}$. The probability that they contradict each other when asked to speak on a fact is

(a) $\frac{7}{20}$ (b) $\frac{3}{20}$ (c) $\frac{1}{5}$ (d) $\frac{4}{5}$.

(ix) Which of the following is **NOT CORRECT**?

(a) An absorbing state is aperiodic (b) Recurrent state is periodic
(c) An ergodic state is recurrent (d) An absorbing state is recurrent.

(x) Moment generating function for the binomial variate $B(10, \frac{1}{4})$ is

(a) $\left(\frac{3}{4}e^t + \frac{1}{4}\right)^{10}$ (b) $\left(\frac{3}{4}e^t - \frac{1}{4}\right)^{10}$
(c) $\left(\frac{1}{4}e^t + \frac{3}{4}\right)^{10}$ (d) $\left(\frac{1}{4}e^t - \frac{3}{4}\right)^{10}$.

Group – B

2. (a) Suppose that it is known that the number of items produced in a factory during a week is a random variable with mean 50.
   (i) What can be said about the probability that this week's production will exceed 75?
   (ii) If the variance of a week's production is known to equal 25, then what can be said about the probability that this week's production will be between 40 and 60? [(MATH3222.1, MATH3222.2) (Apply/IOCQ)]

   (b) Let X be a random variable whose probability density function is given by

$$f_X(x) = \begin{cases} e^{-2x} + \frac{1}{2}e^{-x}, & x > 0 \\ 0, & otherwise \end{cases}.$$

   Find the moment generating function for X. Hence compute the first and second order moments of X about origin.

[(MATH3222.1, MATH3222.2) (Evaluate/HOCQ)]

6 + 6 = 12

3. (a) The number of students who enrol in a course conducted by an institute is a Poisson random variable with mean 100. The institute decided that if the

number of enrolment is 120 or more, the institute will run the course in two separate sections, whereas if fewer than 120 students enrol it will teach all of the students in a single section. Use central limit theorem to find the probability that the institute would make two sections.

[(MATH3222.1 & MATH3222.2) (Apply/IOCQ)]

(b) Let X and Y be two random variables with the joint probability density function:

$$f(x,y) = \begin{cases} 8xy, & 0 < x \le y < 1 \\ 0, & otherwise \end{cases}$$

Find (i) the marginal density function of X and Y.

(ii) the conditional density function of X given $Y = y$ and the conditional density function of Y given $X = x$.

(iii) $P\left(Y \ge \frac{1}{2} \middle| X = \frac{1}{3}\right)$. [(MATH3222.1, MATH3222.2)(Apply/IOCQ)]

6 + 6 = 12

Group – C

4. (a) A student's study habits are as follows: If he studies one night, he is 70% sure not to study the next night. On the other hand, if he does not study one night, he is 60% sure not to study the next night as well. Use Markov chain to find in the long run how often does he study?

[(MATH3222.3 & MATH3222.4) (Evaluate/HOCQ)]

(b) Fit a second degree parabola $y = a + bx + cx^2$ to the following data by the method of least squares.

x	1929	1930	1931	1932	1933	1934	1935	1936	1937
y	352	356	357	358	1360	361	361	360	359

[(MATH3222.3 & MATH3222.4)(Remember/LOCQ)]

4 + 8 = 12

5. (a) Calculate the first four moments about the mean for the following distribution and hence find the coefficient of skewness and kurtosis and comment upon the nature of the distribution.

x	21 − 24	25 − 28	29 − 32	33 − 36	37 − 40	41 − 44
f	40	90	190	110	50	20

[(MATH3222.3, MATH3222.4) (Remember/LOCQ)]

(b) Consider a Markov Chain given by the transition probability matrix

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

Is this Markov chain irreducible? Find all recurrent and transient states. Show your calculation. [(MATH3222.3 & MATH3222.4)(Analyze/IOCQ)]

8 + 4 = 12

Group – D

6. (a) A transmitter has an alphabet of four letters (y_1, y_2, y_3, y_4) and a receiver has an alphabet of three letters (z_1, z_2, z_3). The joint probability matrix is

$$P(Y,Z) = \begin{matrix} & z_1 & z_2 & z_3 \\ y_1 & 0.30 & 0.05 & 0 \\ y_2 & 0 & 0.25 & 0 \\ y_3 & 0 & 0.15 & 0.05 \\ y_4 & 0 & 0.05 & 0.15 \end{matrix}$$

Calculate $H(Y), H(Z), H(Y,Z), H(Y/Z)$ and $H(Z/Y)$.

[(MATH3222.2 & MATH3222.5) (Understand/LOCQ)]

(b) Let the random variable X have three possible outcomes $\{a, b, c\}$. Consider two distributions on this random variable

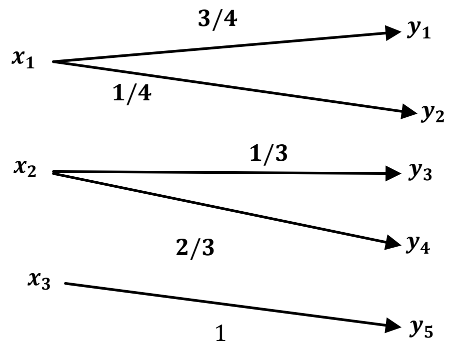
Symbols	$p(x)$	$q(x)$	$r(x)$
a	1/2	1/3	1/6
b	1/4	1/3	1/6
c	1/4	1/3	2/3

Calculate $D(p||q), D(p||r)$ and $D(q||r)$. Are they following any relation?

[(MATH3222.2, MATH3222.5) (Remember/LOCQ)]

7 + 5 = 12

7. (a) A discrete source transmits message x_1, x_2 and x_3 with the probabilities $\frac{1}{2}, \frac{1}{4}$ and $\frac{1}{4}$. The source is connected to the channel given in the following figure



Calculate the mutual information between X and Y.

[(MATH3222.2 & MATH3222.5) (Understand/LOCQ)]

(b) Use Jensen's inequality to find the appropriate inequalities between

(i) $E(e^{-aX})$ and $e^{-aE(X)}$ $, a > 0$

(ii) $E(\sqrt{X})$and $\sqrt{E(X)}$

[(MATH3222.2, MATH3222.5) (Apply/IOCQ)]

6 + 6 = 12

Group – E

8. (a) Let us consider the following joint probability mass function on (X, Y)

$$P(X,Y) = \begin{matrix} & a & b & c \\ 1 & 0.2 & 0.1 & 0 \\ 2 & 0 & 0.1 & 0.1 \\ 3 & 0 & 0.3 & 0.2 \end{matrix}$$

(i) Find the minimum probability of error estimator $\hat{X}(Y)$ and the associated probability.

(ii) Evaluate Fano's inequality for this problem and compare.

[(MATH3222.6) (Evaluate/HOCQ)]

(b) Let

$$X = \begin{cases} 1, & \text{with probability } \frac{1}{3} \\ 2, & \text{with probability } \frac{1}{6} \\ 3, & \text{with probability } \frac{1}{2}. \end{cases}$$

Let $X_1, X_2, \cdots$ be drawn $i.i.d.$ according to this distribution. Find the limiting behaviour of $(X_1 X_2 \ldots X_n)^{\frac{1}{n}}$. [(MATH3222.6)(Apply/IOCQ)]

8 + 4 = 12

9. (a) Consider a sequence of $i.i.d.$ binary random variables, $X_1, X_2, \cdots X_n$, where the probability that $X_i = 1$ is 0.6.

(i) With $n = 25$ and $\in = 0.1$, which sequences fall in the typical set $A_\in^{(n)}$? What is the probability of the typical set? How many elements are there in the typical set?

(ii) How many elements are there in the smallest set that has probability 0.9? (**Use Table-I**)

[(MATH3222.6) (Evaluate/HOCQ)]

(b) Let the transition probability matrix of a two-state Markov chain $X \to Y \to Z$ is given as follows:

$$P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

Verify Data Processing lemma for the above problem. $(i.e. I(X;Z) \leq I(X;Y))$

[(MATH3222.6)(Apply/IOCQ)]

6 + 6 = 12

Cognition Level	LOCQ	IOCQ	HOCQ
Percentage distribution	35.42%	39.58%	25%

Course Outcome (CO):

After the completion of the course students will be able to

MATH3222.1: Articulate the axioms (laws) of probability.

MATH3222.2: Compare and contrast different interpretations of probability theory selecting the preferred one in a specific context.

MATH3222.3: Formulate predictive models to tackle situations where deterministic algorithms are intractable.

MATH3222.4: Quantifies the amount of uncertainty involved in the value of a random variable or the outcome of a random process.

MATH3222.5: Apply the data processing inequality to data science, machine learning and social science.

MATH3222.6: Develop the concept of data compression in the process of encoding information in signal processing.

*LOCQ: Lower Order Cognitive Question; IOCQ: Intermediate Order Cognitive Question; HOCQ: Higher Order Cognitive Question

Table-I

k	$\binom{n}{k}$	$\binom{n}{k} p^k (1-p)^{(n-k)}$	Distribution Function	$p(x^n) = p^k (1-p)^{(n-k)}$	$-\frac{1}{n} log_2 p(x^n)$
0	1	1.1259E-10	1.1259E-10	1.1259E-10	1.321928095
1	25	4.22212E-09	4.22212E-09	1.68885E-10	1.298529595
2	300	7.59982E-08	8.02204E-08	2.53327E-10	1.275131095
3	2300	8.7398E-07	9.542E-07	3.79991E-10	1.251732595
4	12650	7.21033E-06	8.16453E-06	5.69987E-10	1.228334095
5	53130	4.54251E-05	5.35896E-05	8.5498E-10	1.204935595
6	177100	0.000227126	0.000280715	1.28247E-09	1.181537095
7	480700	0.000924725	0.00120544	1.92371E-09	1.158138595
8	1081575	0.003120948	0.004326388	2.88556E-09	1.134740095
9	2042975	0.008842685	0.013169073	4.32834E-09	1.111341595
10	3268760	0.021222445	0.034391518	6.49251E-09	1.087943095
11	4457400	0.043409546	0.077801064	9.73876E-09	1.064544595
12	5200300	0.075966705	0.153767769	1.46081E-08	1.041146095
13	5200300	0.113950058	0.267717827	2.19122E-08	1.017747595
14	4457400	0.146507217	0.414225044	3.28683E-08	0.994349094
15	3268760	0.161157939	0.575382982	4.93025E-08	0.970950594
16	2042975	0.151085568	0.72646855	7.39537E-08	0.947552094
17	1081575	0.119979715	0.846448265	1.10931E-07	0.924153594
18	480700	0.079986477	0.926434742	1.66396E-07	0.900755094
19	177100	0.044203053	0.970637795	2.49594E-07	0.877356594
20	53130	0.019891374	0.990529169	3.74391E-07	0.853958094
21	12650	0.007104062	0.997633231	5.61586E-07	0.830559594
22	2300	0.001937471	0.999570703	8.42379E-07	0.807161094
23	300	0.000379071	0.999949773	1.26357E-06	0.783762594
24	25	4.73838E-05	0.999997157	1.89535E-06	0.760364094
25	1	2.84303E-06	1	2.84303E-06	0.736965594