ALGEBRAIC STRUCTURES
(MATH 2201)

Time Allotted : 3 hrs. **Full Marks : 70**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and
any 5 (five) from Group B to E, taking at least one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A
(Multiple Choice Type Questions)

1. Choose the correct alternative for the following: **10 × 1 = 10**

(i) If $\phi: x \rightarrow \{the\ highest\ prime\ factor\ of\ x\}$ and domain of f is $\{12, 14, 17, 22\}$, then range set of f is
(a) $\{3, 2, 17, 11\}$ (b) $\{3, 7, 17, 2\}$
(c) $\{6, 7, 17, 11\}$ (d) $\{3, 7, 17, 11\}$.

(ii) $Z_n = \{0,1,2,3, \dots, n-1\}$ is a group under multiplication if $n =$
(a) 6 (b) 10 (c) 12 (d) 7.

(iii) On the set $A = \{1,2,3\}$ the relation $R = \{(2,1), (1,2), (3,3)\}$ is
(a) Reflexive (b) Symmetric
(c) Transitive (d) All of the above.

(iv) Let $X = \{2,5,9\}$, $Y = \{1,2,7\}$. The mapping $f: X \rightarrow Y$ defined as $f\{(2,1), (5,1), (9,2)\}$ is
(a) One – to – one (b) Many to one
(c) Onto (d) One – one.

(v) If S and T are two subgroups of a group G. Then which of the following is a subgroup?
(a) $S \cup T$ (b) $S \cap T$ (c) $S - T$ (d) $G - S$

(vi) Consider the poset $S = \{1, 2, 3, 4, 6, 9\}$ with respect to divisibility. The maximal element of S is/are
(a) 9 (b) 6 (c) 6, 9 (d) 4, 6, 9.

(vii) The operation $*$ defined by $a * b = \frac{a+b}{5}$ is a binary relation on
(a) set of integers (b) set of positive integers
(c) set of natural numbers (d) set of rational numbers.

(viii) If $f: G \rightarrow G'$ be a homomorphism then the kernel of f is
(a) a subgroup of G (b) a subgroup of G'
(c) a subset of G (d) null set.

(ix) Let $G = (\mathbb{Z}, +)$. The mapping $\phi: G \to G$ defined by $f(x) = x + 2$ is
(a) homomorphism (b) not homomorphism
(c) automorphism (d) isomorphism.

(x) Consider the set $\mathbb{Z}_4 = \{[0],[1],[2],[3]\}$, the set of all residue classes modulo 4. The residue class [1] contains the elements
(a) $\{0,4,8,\dots\}$ (b) $\{0,1\}$
(c) $\{1,5,9,\dots\}$ (d) $\{0,5,7,\dots\}$.

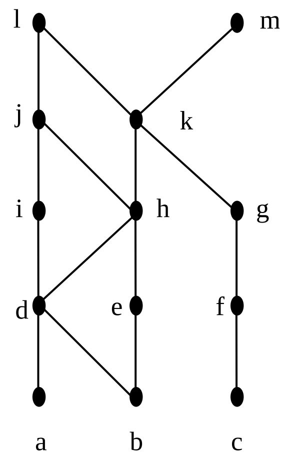
Group – B

2. (a) Let $A = \{a, b, c\}$ and let R and S be relations on A whose matrices are $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $M_S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. Find $S \circ R$. Is the relation reflexive, symmetric, anti-symmetric, or transitive? Justify your answer.
[(MATH2201.1, MATH2201.5)(Understand/LOCQ)]

(b) Let $S = \mathbb{N} \times \mathbb{N}$. Define the relation $\sim$ on S where $(a,b) \sim (c,d)$ if and only if $a + d = b + c$. Prove that $\sim$ is an equivalence relation.
[(MATH2201.1, MATH2201.5)(Evaluate/HOCQ)]

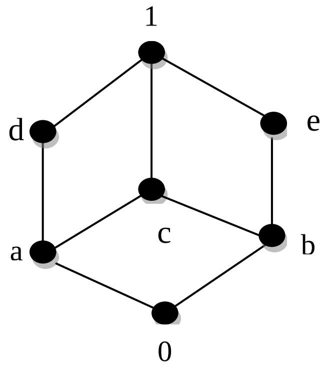
6 + 6 = 12

3. (a) For the Hasse diagram given below, find maximal, minimal, greatest, least, LB, glb, UB, lub for the subsets:
(i) $\{d, k, f\}$ (ii) $\{b, h, f\}$ (iii) $\{d\}$ (iv) $\{l, m\}$



[(MATH2201.1, MATH2201.5)(Evaluate/HOCQ)]

(b) Is the following lattice a distributive lattice? Justify your answer.



[(MATH2201.1, MATH2201.5) (Analyze/IOCQ)]

6 + 6 = 12

Group – C

4. (a) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 5 \end{pmatrix}$
Compute each of the following: (i) α^{-1} (ii) $\beta\alpha$ (iii) $\alpha\beta$
[(MATH2201.2, MATH2201.3, MATH2201.4, MATH2201.6)(Remember/LOCQ)]

(b) Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix}$ be two permutations. Show that $AB \neq BA$.
[(MATH2201.2, MATH2201.3, MATH2201.4, MATH2201.6) (Remember/LOCQ)]

(c) Show that the symmetric group S_3 is non-commutative.
[(MATH2201.2, MATH2201.3, MATH2201.4, MATH2201.6)(Analyse/IOCQ)]

6 + 3 + 3 = 12

5. (a) If G is a group of even order, prove that it has an element $a \neq e$ satisfying $a^2 = e$.
[(MATH2201.2, MATH2201.3, MATH2201.4, MATH2201.6)(Create/HOCQ)]

(b) Let A_n denote the alternating group of degree n, i.e., the group of even permutation of $\{1,2,3, \dots, n\}$. Prove that A_n has $\frac{n!}{2}$ elements, for $n > 1$.
[(MATH2201.2, MATH2201.3, MATH2201.4, MATH2201.6)(Analyze/IOCQ)]

6 + 6 = 12

Group – D

6. (a) Show that the set $G = \{1, 2, 3, 4, 5, 6\}$ forms a cyclic group under the operation multiplication modulo 7. Find all generators of this group.
[(MATH2201.2, MATH2201.3, MATH2201.4, MATH2201.6) (Create/HOCQ)]

(b) Consider $\mathbb{Z}$, the group of integers under addition. Let H be a subset of $\mathbb{Z}$ consisting of all multiples of a positive integer m, i.e. , km, $(k = 0, \pm 1, \pm 2, \dots \dots)$. Show that H is a subgroup of $\mathbb{Z}$.
[(MATH2201.2, MATH2201.3, MATH2201.4, MATH2201.6) (Apply/IOCQ)]

(c) If x be an element of a group $(G, *)$ and $o(x) = 20$, find the order of x^8.
[(MATH2201.2, MATH2201.3, MATH2201.4, MATH2201.6)(Analyse/IOCQ)]

6 + 3 + 3 = 12

7. (a) Show that every proper subgroup of a group of order 6 is cyclic.
[(MATH2201.2, MATH2201.3, MATH2201.4, MATH2201.6) (Apply/IOCQ)]

(b) Prove that the set of matrices $H = \left\{ \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \right\}$ is a normal subgroup of the group of all non-singular 2×2 matrices.
[(MATH2201.2, MATH2201.3, MATH2201.4, MATH2201.6)(Evaluate/HOCQ)]

6 + 6 = 12

Group – E

8. (a) Define a map $\phi: \mathbb{C} \to M_2(\mathbb{R})$ by $\varphi(a + bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Show that ϕ is an isomorphism of $\mathbb{C}$ with its image in $M_2(\mathbb{R})$.
[(MATH2201.2, MATH2201.3, MATH2201.4) (Analyse/IOCQ)]

(b) Define an idempotent element. Prove that the only idempotents in an integral domain are 0 and 1.

[(MATH2201.2, MATH2201.3, MATH2201.4) (Understand/LOCQ)]

7 + 5 = 12

9. (a) Let R be the ring of 2×2 matrices of the form $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$, where $a, b \in \mathbb{R}$. Show that although R is a ring that has no identity, we can find a subring S of R with an identity.

[(MATH2201.2, MATH2201.3, MATH2201.4) (Analyse/IOCQ)]

(b) Let $(F, +, .)$ be a field and $a, b \in F$ with $b \neq 0$. Then show that $a = 1$ when $(ab)^2 = ab^2 + bab - b^2$.

[(MATH2201.2, MATH2201.3, MATH2201.4) (Evaluate/HOCQ)]

6 + 6 = 12

Cognition Level	LOCQ	IOCQ	HOCQ
Percentage distribution	20.83%	41.67%	37.5%

Course Outcome (CO):

After the completion of the course students will be able to

MATH2201.1 Describe the basic foundation of computer related concepts like sets, POsets, lattice and Boolean Algebra.

MATH2201.2 Analyze sets with binary operations and identify their structures of algebraic nature such as groups, rings and fields.

MATH2201.3 Give examples of groups, rings, subgroups, cyclic groups, homomorphism and isomorphism, integral domains, skew-fields and fields.

MATH2201.4 Compare even permutations and odd permutations, abelian and non-abelian groups, normal and non-normal subgroups and units and zero divisors in rings.

MATH2201.5 Adapt algebraic thinking to design programming languages.

MATH2201.6 Identify the application of finite group theory in cryptography and coding theory.

*LOCQ: Lower Order Cognitive Question; IOCQ: Intermediate Order Cognitive Question; HOCQ: Higher Order Cognitive Question