LINEAR ALGEBRA
(MATH 4182)

Time Allotted : 3 hrs **Full Marks : 70**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 5 (five) from Group B to E, taking at least one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A
(Multiple Choice Type Questions)

1. Choose the correct alternative for the following: **10 × 1 = 10**

(i) The eigenvalues of the matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{pmatrix}$ are

(a) $(0, -1, -3)$ (b) $(0, -2, -3)$ (c) $(0, 2, 3)$ (d) $(0, 1, 3)$.

(ii) The number of linearly independent eigenvectors of the matrix $B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is

(a) 1 (b) 2 (c) 3 (d) 0.

(iii) The dimension of the vector space of all 2×2 matrices such that $a_{11} + a_{22} = 0$ is

(a) 2 (b) 3 (c) 4 (d) 5.

(iv) Which of the following is not always true?

(a) The determinant of a positive definite matrix is positive
(b) A positive definite matrix is invertible
(c) A diagonal matrix with positive diagonal entries is positive definite
(d) A symmetric matrix with positive determinant is positive definite.

(v) If $\|.\|$ represents the usual norm in $\mathbb{R}^n$, $\|v\|$ when $v = (4, -2, 2, 1)$ is given by

(a) 25 (b) 5 (c) $\sqrt{5}$ (d) $5\sqrt{5}$.

(vi) When the matrix equation $Ax = b$ is inconsistent and p is the projection of b onto the column space of A, then the least-squares solution of $Ax = b$ minimizes

(a) $\|b - Ax\|$ (b) $\|p - Ax\|$ (c) $\|A^T Ax\|$ (d) $\|Ax\|$.

(vii) Which of the following sets form a basis of $\mathbb{R}^3$?

(a) $\{(0,1,0), (-1,0,0), (3,2,0)\}$ (b) $\{(0,0,3), (0,2,3), (0,-1,0)\}$
(c) $\{(3,0,0), (0,-2,0), (0,0,5)\}$ (d) $\{(3,0,-2), (1,0,0), (0,0,1)\}$.

(viii) The kernel of the linear transformation $T: U \to V$ is
(a) subspace of U (b) subgroup of U
(c) subset of U (d) subspace of V.

(ix) The intersection of the Range and Null space of T is the
(a) $\dim(T)$ (b) $\operatorname{rank}(T)$
(c) zero sub-space of V (d) $\operatorname{nullity}(T)$.

(x) Let V be the vector space of functions with basis $S = \{\sin t, \cos t, e^t\}$ and let D be the usual differential operator on V defined by $D(f(t)) = \frac{d}{dt}(f(t))$. Then the matrix representing D in the basis S is given by

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Group – B

2. (a) Find the eigenvalues and eigenvectors of the matrix given by $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$. Determine the algebraic multiplicity and geometric multiplicity of each of the eigenvalues.

(b) Diagonalize the following matrix and hence find A^6, where $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$.

6 + 6 = 12

3. (a) Find the singular value decomposition (SVD) of the matrix given by $A = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}$.

(b) Let $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$. Find a "real cube root" of A, that is, find a matrix B having real eigenvalues such that $B^3 = A$.

6 + 6 = 12

Group – C

4. (a) (i) Investigate if $S = \{(x_1, x_2 + 1, x_3, x_4) | x_1, x_2, x_3, x_4 \in \mathbb{R}\}$ forms a subspace of $\mathbb{R}^4$.

(ii) Let V be the set of all ordered pairs (x, y) in $\mathbb{R}^2$ with vector addition defined as $(x_1, y_1) + (x_2, y_2) = (2x_1 - 3x_2, y_1 - y_2)$ and scalar multiplication defined as $\alpha(x_1, y_1) = \left(\frac{\alpha x_1}{3}, \frac{\alpha y_1}{3}\right)$. Discuss whether V is a vector space or not.

(b) Show whether the given sets of vectors are identical or not:
$S = span\{(1,2,-1), (-3,0,0)\}, T = span\{(1,0,0), (1,3,0)\}$.

(3 + 3) + 6 = 12

5. (a) Find the dimension and basis of the vector space spanned by the set of vectors $S = \{(1,3,5), (2,-1,4), (-2,8,2)\}$.

(b) Determine whether the vectors $v_1 = (1,1,-2,-2), v_2 = (2,-3,0,2)$,

$v_3 = (-2, 0, 2, 2), v_4 = (3, -3, -2, 2)$ are linearly dependent or independent in $\mathbb{R}^4$.

6 + 6 = 12

Group – C

6. (a) Consider $f(t) = 3t - 5$ and $g(t) = t^2$ in the polynomial space $P(t)$ with inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)dt, \forall f, g \in P(t)$. Find
(i) $\langle f, g\rangle$
(ii) $\|f\|$ and $\|g\|$.

(b) Consider the vector space $P(t)$ with inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)dt$. Apply Gram-Schmidt algorithm to the set $\{1, t, t^2\}$ to obtain an orthogonal set with integer coefficients.

(2 + 4) + 6 = 12

7. (a) Find the QR factorization of the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

(b) Find an orthogonal matrix P whose first row is $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$.

6 + 6 = 12

Group – E

8. (a) Determine whether the transformations given below are linear transformations or not:
(i) $T: \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(x, y) = (0, 0, 0)$
(ii) $T: \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(x, y) = (1, 1, 1)$.

(b) Let $f: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $f(x, y, z) = (x + 2y - z, y + z, x + y - 2z)$. Find a basis and the dimension of (i) the image of f, and (ii) the kernel of f.

6 + (3 + 3) = 12

9. (a) Verify the rank-nullity-dimension theorem for the linear transformation defined by $f(x, y, z, w) = (x - y + z + w, x + 2z - w, x + y + 3z - 3w)$.

(b) Let T be a linear operator in $\mathbb{R}^3$ defined by
$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$.
Find the matrix of T in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3\}$ where $\alpha_1 = (1, 0, 1)$, $\alpha_2 = (-1, 2, 1)$ and $\alpha_3 = (2, 1, 1)$.

6 + 6 = 12

Department & Section	Submission link:
AEIE	https://classroom.google.com/c/MTE5MjY3MzE1OTQ5/a/MjcxNDc4NTc0NjYz/details

CSE A	https://classroom.google.com/c/MTQxMTE3NTM4MTM2/a/MjcxNDc4NTc0NTQ5/details
CSE B	https://classroom.google.com/c/MTE5MjY3MzE1OTQ5/a/MjcxNDc4NTc0NjIx/details
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