LINEAR ALGEBRA
(MATH 4126)

Time Allotted : 3 hrs **Full Marks : 70**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 5 (five) from Group B to E, taking at least one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A
(Multiple Choice Type Questions)

1. Choose the correct alternative for the following: **10 × 1 = 10**

(i) If A be a singular matrix then an eigenvalue of A is
(a) 1 (b) 2 (c) -1 (d) 0

(ii) The geometric multiplicity of the eigenvalue 0 of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ is
(a) 0 (b) 1 (c) 2 (d) 3

(iii) If A is a 3×3 diagonalisable matrix, then the number of linearly independent eigenvectors of A is
(a) 1 (b) 2 (c) 3 (d) 4

(iv) The linear map $T: \mathbb{R}^4 \to \mathbb{R}^4$ of the matrix $M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ with respect to standard basis is
(a) $T(x, y, z, w) = (z, y, x, w)$ (b) $T(x, y, z, w) = (0, x, w, y)$
(c) $T(x, y, z, w) = (z, x, w, y)$ (d) $T(x, y, z, w) = (x - y, 0, y, w)$

(v) Let V be a finite dimensional vector space and $f: V \to V$ be linear. Then which of the following is true?
(a) $\dim V = \text{rank } f$ (b) $\dim V = \dim(\text{Ker } f)$
(c) $\dim V = \dim(\text{Im } f)$ (d) $\dim V = \dim(\text{Ker } f) + \dim(\text{Im } f)$

(vi) Consider the following vectors in $\mathbb{R}^3$: $u_1 = (1,2,1), u_2 = (2,1,-4), u_3 = (3,-2,1)$. Which of the following statement is true?
(a) u_1 is parallel to u_2
(b) u_2 is parallel to u_3
(c) all of them are orthogonal to each other
(d) only u_1 and u_2 are orthogonal to each other but u_3 is not.

(vii) If $u = (1, -2, k)$ is a linear combination of $(3,0,-2)$ and $(2,-1,-5)$ then the value of k is

(a) 2 (b) -5 (c) 1 (d) -8

(viii) The dimension of a vector space is

(a) the number of vectors in a basis
(b) the number of subspaces
(c) the dimension of its subspaces
(d) the number of vectors in a spanning subset

(ix) Which of the following sets is a basis for the subspace $W = \left\{\begin{bmatrix} x & y \\ 0 & t \end{bmatrix} : x + 2y + t = 0, y + t = 0\right\}$ of the vector space of all real 2×2 matrices?

(a) $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ (b) $\left\{\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\right\}$

(c) $\left\{\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}\right\}$ (d) $\left\{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\right\}$

(x) The real quadratic form in two variables a, b associated with the matrix $\begin{pmatrix} 5 & 1 \\ 1 & -1 \end{pmatrix}$ is

(a) $5a^2 + 2ab - b^2$ (b) $5a^2 - 2ab + b^2$ (c) $a^2 + 5ab - b^2$ (d) $a^2 - 5ab + b^2$

Group – B

2. (a) Let $A = \begin{pmatrix} 7 & 3 \\ 3 & -1 \end{pmatrix}$. Find an orthogonal matrix P such that $D = P^{-1}AP$.
[(MATH4126.1) (Apply/IOCQ)]

(b) Find the Singular Value Decomposition of $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$.
[(MATH4126.1) (Evaluate/HOCQ)]

5 + 7 = 12

3. (a) A is a 3×3 real matrix having eigenvalues 2,3 and 1. $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ are the eigenvectors of A corresponding to the eigenvalues 2,3 and 1 respectively. Find the matrix A. [(MATH4126.1) (Understand/LOCQ)]

(b) Let $q(x, y) = 2x^2 - 4xy + 5y^2$. Find an orthogonal substitution that diagonalizes q. [(MATH4126.1) (Evaluate/HOCQ)]

4 + 8 = 12

Group – C

4. (a) Let V be the vector space $\mathbb{R}^3(\mathbb{R})$ and let $W_1 = \{(0, y, z) : y, z \in \mathbb{R}\}$ and $W_2 = \{(x, y, 0) : x, y \in \mathbb{R}\}$.

(i) Find $W_1 \cap W_2$. Is it a subspace of V? Justify.
(ii) Find $W_1 \cup W_2$. Is it a subspace of V? Justify.

[(MATH4126.2) (Understand/LOCQ)]

(b) Find the conditions on (x, y, z) such that it belongs to the span of (2, 1, 0), (1, -1, 2) and (0, 3, -4). [(MATH4126.2) (Apply/IOCQ)]

(3 + 3) + 6 = 12

5. (a) Find the possible values of k for which the vectors (1, -1, 2), (0, k, 3) and (-1, 2, 3) are linearly dependent. [(MATH4126.2) (Understand/LOCQ)]

(b) Let V be the vector space of 2×2 symmetric matrices over $\mathbb{R}$. Show that dim (V) = 3 by constructing a basis for V. [(MATH4126.2) (Evaluate/HOCQ)]

5 + 7 = 12

Group – D

6. (a) Apply the Gram-Schmidt orthogonalization process to find an orthonormal basis for the subspace U of $\mathbb{R}^4$ spanned by $S = \{(1,1,1,1), (1,2,4,5), (1,-3,-4,-2)\}$. [(MATH4126.3) (Apply/IOCQ)]

(b) Find $\cos\theta$ where θ is the angle between $A = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ where $\langle A, B \rangle = trace(B^T A)$. [(MATH4126.3) (Evaluate/HOCQ)]

6 + 6 = 12

7. (a) Prove that an orthogonal set of nonzero vectors in a Euclidean space V is linearly independent. [(MATH4126.3) (Apply/IOCQ)]

(b) Let S consist of the following vectors in $\mathbb{R}^4$:
$u_1 = (1,1,0,-1), u_2 = (1,2,1,3), u_3 = (1,1,-9,2), u_4 = (16,-13,1,3)$.
Show that S is orthogonal and a basis of $\mathbb{R}^4$. [(MATH4126.3) (Evaluate/HOCQ)]

6 + 6 = 12

Group – E

8. (a) Let V be a vector space of n-square real matrices. Let M be an arbitrary but fixed matrix in V. Let $T: V \to V$ be defined by $T(A) = AM + MA$, where A is any matrix in V. Examine if T is linear. [(CO4MATH4126.5)(Understand/LOCQ)]

(b) Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear mapping. Then prove that $Ker\, T = \{\alpha \in V : T(\alpha) = \theta'\}, \theta'$ is the zero vector in W, is a subspace of V. [(MATH4126.5) (Apply/IOCQ)]

6 + 6 = 12

9. (a) Let $P_2(\mathbb{R})$ be the vector space over $\mathbb{R}$ consisting of all polynomials with real coefficients of degree 2 or, less. Let $B = \{1, x, x^2\}$ be a basis of the vector space $P_2(\mathbb{R})$. For each linear transformation $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined below, find the matrix representation of T with respect to the basis B. For, $f(x) \in P_2(\mathbb{R})$, define T as follows:

(i) $T(f(x)) = \frac{d^2}{dx^2} f(x) - 3\frac{d}{dx} f(x)$

(ii) $T(f(x)) = \int_{-1}^{1} (t-x)^2 f(t)dt$ [(MATH4126.5) (Apply/IOCQ)]

(b) Determine the linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ which maps the basis vectors $\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$ to $\{(1,1,1), (1,1,1), (1,1,1)\}$ of $\mathbb{R}^3$, respectively. [(MATH4126.5) (Evaluate/HOCQ)]

(3 + 5) + 4 = 12

Cognition Level	LOCQ	IOCQ	HOCQ
Percentage distribution	22%	38%	40%

Course Outcome (CO):

After the completion of the course students will be able to

MATH4126. 1. Explain concepts of diagonalization, orthogonal diagonalization and Singular Value Decomposition (SVD).

MATH4126. 2. Discuss basis, dimension and spanning sets.

MATH4126. 3. Design Gram-Schmidt Orthogonalization Process and QR decomposition using concepts of inner product spaces.

MATH4126. 4. Analyze Least squares solutions to find the closest line by understanding projections.

MATH4126. 5. Define linear transformations and change of basis.

MATH4126. 6. Illustrate applications of SVD such as, Image processing and EOF analysis, applications of Linear algebra in engineering with graphs and networks, Markov matrices, Fourier matrix, Fast Fourier Transform and linear programming.

*LOCQ: Lower Order Cognitive Question; IOCQ: Intermediate Order Cognitive Question; HOCQ: Higher Order Cognitive Question

Department & Section	Submission link:
AEIE	https://classroom.google.com/c/NDA0OTk1NzkyMDAw/a/NDYzOTQzNzk5NTk3/details
CSE	https://classroom.google.com/c/NDA0OTk1NzkyMDAw/a/NDYzOTQzNzk5NTk3/details