# ALGEBRAIC STRUCTURES
# (MATH 2201)

**Time Allotted : 3 hrs** **Full Marks : 70**

***Figures out of the right margin indicate full marks.***

***Candidates are required to answer Group A and any 5 (five) from Group B to E, taking at least one from each group.***

***Candidates are required to give answer in their own words as far as practicable.***

## Group – A
## (Multiple Choice Type Questions)

1. Choose the correct alternative for the following: **10 × 1 = 10**

(i) Let $S$ be a set of $n$ elements. The number of ordered pairs in the largest equivalence relation on $S$ is
(a) 1 (b) $n$ (c) $n-1$ (d) $n^2$.

(ii) Consider the binary relation $R = \{(x,y),(x,z),(z,x),(z,y)\}$ on the set $\{x,y,z\}$. Which one of the following is TRUE?
(a) $R$ is symmetric but not antisymmetric
(b) $R$ is not symmetric but antisymmetric
(c) $R$ is both symmetric and antisymmetric
(d) $R$ is neither symmetric nor antisymmetric.

(iii) Which one of the following is true?
(a) $R \cap S = (R \cup S) - [(R-S) \cup (S-R)]$
(b) $R \cup S = (R \cap S) - [(R-S) \cup (S-R)]$
(c) $R \cap S = (R \cup S) - [(R-S) \cap (S-R)]$
(d) $R \cap S = (R \cup S) \cup (R-S)$.

(iv) In $\mathbb{Z}_7$,
(a) $\overline{7} = \overline{15}$ (b) $\overline{7} = \overline{48}$ (c) $\overline{7} = \overline{70}$ (d) $\overline{7} = \overline{1}$.

(v) If $x$ is an element of a group $G$ and $O(x) = 5$, then
(a) $O(x^{10}) = 5$ (b) $O(x^{15}) = 5$
(c) $O(x^{23}) = 5$ (d) $O(x^{20}) = 5$.

(vi) Index of a subgroup $H$ of a group $G$ is 5 and its order is 3. The order of the group $G$ is
(a) 8 (b) 10 (c) 15 (d) 25.

(vii) In the additive group $\langle \mathbb{R}, + \rangle$, $(10)^0 =$
(a) 1 (b) 0 (c) $-1$ (d) 10.

(viii) $H = \{1, -1\}$ is a multiplicative subgroup of $G = \{1, -1, i, -i\}$. Then the index of $H$ in $G$ is:
(a) 1 (b) 2 (c) 3 (d) 4.

(ix) Which one of the following rings is an integral domain?
(a) $\mathbb{Z}_{101}$ (b) $\mathbb{Z}_{200}$ (c) $\mathbb{Z}_{2001}$ (d) $\mathbb{Z}_{356}$

(x) The unity element of the ring of matrices $\left\{\begin{pmatrix} x & y \\ -y & x \end{pmatrix} : x, y \in \mathbb{R}\right\}$ with respect to matrix addition and matrix multiplication is
(a) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

## Group – B

2. (a) Let $A = \{a, b, c, d\}$, $B = \{x, y, z\}$, $C = \{p, q, r, s\}$, $\rho = \{(a, y), (b, x), (c, z), (d, x)\}$ and $\sigma = \{(x, q), (x, r), (y, s), (z, p)\}$. Find the relation matrix $M_\rho$, $M_\sigma$ and $M_{\sigma\circ\rho}$.

(b) Find whether the following relation $\rho$ defined by "$a\rho b$ iff $|a| \le b$", $a, b \in \mathbb{R}$, the set of real numbers, is (i) reflexive, (ii) symmetric, (iii) transitive.

**6 + 6 = 12**

3. (a) Draw the Hasse diagram representing the partial ordering $\{(a, b) |\, a \text{ divides } b\}$ on the set $S = \{1,2,3,4,6,8,12\}$. Is the poset a lattice? Justify your answer.

(b) Define a distributive lattice. Prove that a lattice $(L, \le, \vee, \wedge)$is distributive if and only if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \forall\, a, b, c \in L$.

**5 + (2 + 5) = 12**

## Group – C

4. (a) Show that the set $G = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a group with respect to addition, where $\mathbb{Q}$ denotes the set of rational numbers.

(b) Let $G$ be a group and $a, b \in G$. Show that $(aba^{-1})^n = aba^{-1}$ if and only if $b = b^n$.

(c) Let $S$ be a set having $n$ elements. How many distinct binary operations can be defined on $S$ ? Explain your answer.

**4 + 6 + 2 = 12**

5. (a) If $a$ is a permutation given by $(1\ 2\ 3\ 4)$, then show that the set $\{a, a^2, a^3, a^4\}$ forms a cyclic group.

(b) Find a solution of the equation $ax = b$ in $S_3$, where $a = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$.

(c) Let $G$ be a group. If $a, b \in G$, then prove that $(ab)^{-1} = b^{-1}a^{-1}$.

**6 + 3 + 3 = 12**

## Group – D

6. (a) Prove that the intersection of any two subgroups of a finite group $G$ is a subgroup of $G$. Is a similar result true for union? Justify your answer.

   (b) State and prove Lagrange's theorem regarding the order of a subgroup of a finite group.

**(4 + 2) + 6 = 12**

7. (a) If $a$ be an element of order $n$ in a group $G$ and $p$ be prime to $n$, then prove that $a^p$ is also of order $n$.

   (b) Show that every proper subgroup of a group of order 6 is cyclic.

**6 + 6 = 12**

## Group – E

8. (a) Find all the idempotent elements in the ring $\mathbb{Z}_6 \times \mathbb{Z}_{12}$.

   (b) Determine whether the given map $\phi: GL(2,\mathbb{R}) \longrightarrow (\mathbb{R},+)$ defined by $\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + b + c + d$ is a homomorphism, where $GL(2,\mathbb{R})$ is the general linear group of degree 2.

**7 + 5 = 12**

9. (a) Examine if the ring of matrices $\left\{\begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a, b \in \mathbb{R}\right\}$ is a field.

   (b) Prove that the characteristic of an integral domain is either zero or prime.

**6 + 6 = 12**

| Department & Section | Submission Link |
|---|---|
| CSE-A | https://classroom.google.com/c/MzI3NTY0OTc0MDM3/a/MzczNDM0MzY0NDUx/details |
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