ADVANCED PROBABILITY AND INFORMATION THEORY (MATH 3222)

Time Allotted : 3 hrs **Full Marks : 70**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and <u>any 5 (five)</u> from Group B to E, taking <u>at least one</u> from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A
(Multiple Choice Type Questions)

1. Choose the correct alternative for the following: **10 × 1 = 10**

(i) The probability of having at least one six from 3 throws of a perfect die is
(a) $\left(\frac{5}{6}\right)^3$ (b) $1-\left(\frac{5}{6}\right)^3$ (c) $\left(\frac{1}{6}\right)^3$ (d) $1-\left(\frac{1}{6}\right)^3$.

(ii) If $M_X(t)$ be the moment generating function of a random variable$X$, then that of $Y=-5X+1$ is
(a) $e^t M_X(5t)$ (b) $e^{-t} M_X(5t)$
(c) $e^t M_X(-5t)$ (d) $e^{-t} M_X(-5t)$.

(iii) Which **one** of the following inequality is **correct**?
(a) $P(X \geq k) \geq \frac{E(X)}{k}, \quad k>0$ (b) $P(X \geq k) \leq \frac{E(X)}{k}, \quad k>0$
(c) $P(X \geq k) \geq \frac{Var\,(X)}{k}, \quad k>0$ (d) $P(X \geq k) \leq \frac{E(X)}{k}$, for any $k \in \mathbb{R}$.

(iv) The second and forth moments about mean are 4 and 48 respectively, then the distribution is:
(a) Leptokurtic (b) Platykurtic
(c) Mesokurtic (d) Positively skewed.

(v) If $P = \begin{array}{c} 0 \\ 1 \\ 2 \end{array}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$ is a possible transition probability matrix with state space $\{0, 1, 2\}$, then the state $\{0\}$ is
(a) Recurrentstate (b) Transient state
(c) Recurrent state with period 2 (d) absorbing states.

(vi) If X and Y are independent random variable, then joint entropy $H(X,Y)$ of (X,Y) is
(a) $H(X)H(Y)$ (b) $H(X)$ (c) $H(X)+H(Y)$ (d) $H(Y)$.

(vii) The mutual information $I(X;Y)$ is

(a) $H(X) - H(Y/X)$ (b) $H(Y) - H(Y/X)$

(c) $H(X) + H(X/Y)$ (d) $H(Y) + H(X/Y)$.

(viii) Which **one** of the following inequality is **correct**? (Where $E(X)$ denote the expectation of X)

(a) $E(e^{-aX}) \geq e^{-aE(X)},\ a > 0$ (b) $\sqrt{E(X)} \leq E(\sqrt{X})$

(c) $E(X^2) \leq (E(X))^2$ (d) $E(|X|) \leq |E(X)|$.

(ix) The number of elements in the typical set $A_{\epsilon}^{(n)}$ is nearly

(a) $2^{-n(H+\epsilon)}$ (b) $H^{2n}+\epsilon$ (c) $n^{2(H-\epsilon)}$ (d) $2^{n(H+\epsilon)}$.

(x) The joint probability density function of the random variable X and Y is given by

$$f(x,y) = \begin{cases} ce^{-(x+y)}, & 0 < y < x < \infty \\ 0, & otherwise \end{cases}$$

Then the value of c is

(a) 1 (b) 2 (c) 5 (d) 7 .

Group – B

2. (a) The guaranteed average life of a certain type of electric light bulb is 1000h with a standard deviation 125h. It is decided to sample the output so as to ensure that 90% of the bulbs do not fall short of the guaranteed average by more than 2.5%. Use Central Limit Theorem to find the minimum sample size.

(b) Find moment generating function to evaluate mean and variance of a random variable X which follows normal distribution with parameters μ and σ.

6 + 6 = 12

3. (a) A random variable X has the probability density function $f(x) = \begin{cases} 12x^2(1-x), 0 < x < 1 \\ 0, \quad elesewhere \end{cases}$. Compute $P(|X - \mu| \geq 2\sigma)$, where μ and σ are respectively mean and standard deviation for X. Also compare it with the limit given by Chebyshev's inequality.

(b) One bag contains three identical cards marked 1, 2, 3 and another bag contains two cards marked 1, 2. Two cards are randomly chosen one from each bag and the numbers observed. Denote the minimum of the two by X and the sum of the two by Y. Find the joint probability mass function of (X,Y) and the marginal probability mass function of X and Y. Also find $P(|X - Y| \geq 2)$ and $P(X = 1/Y \leq 4)$.

6 + 6 = 12

Group – C

4. (a) A salesman's territory consist of 3 cities A, B and C. He never sells in the same city on the successive days. If he sells in city A then the next day he sells in

city B. However if he sells in either B or C, then the next day he is twice as likely to sell in city A as in other city. In the long run, how often does he sell in each of the cities? If he sells in the city C on Monday in a particular week, then what is the probability that he sells again in city C on Wednesday in the same week?

(b) Find a second degree parabola $y = a + bx + cx^2$ to fit given data, using the method of least squares:

x	1	2	3	4	5	6	7	8
y	2	6	7	8	10	11	11	9

6 + 6 = 12

5\. (a) Calculate the first four moments about the mean for the following distribution and hence find the coefficient of skewness and kurtosis and comment upon the nature of the distribution.

Class Interval	35-40	40-45	45-50	50-55	55-60	60-65	65-70
Frequency	6	8	17	21	15	11	2

(b) The following table provides data about the percentage of students who have free university meals and their CGPA scores. Calculate the Spearman's Rank Correlation between the two and interpret the result.

State University	Pune	Chennai	Delhi	Kanpur	Goa	Indore	Guwahati
% of students having free meals	14.4	7.2	27.5	33.8	38.0	15.9	4.9
% of students scoring above 8.5 CGPA	54	64	44	32	37	68	62

8 + 4 = 12

Group – D

6\. (a) A discrete source transmits messages x_1, x_2, x_3 with probabilities $p(x_1) = 0.3$, $p(x_2) = 0.25$ and $p(x_3) = 0.45$. The source is connected to the channel whose conditional probability matrix is

$$P\left({}^{Y}/_{X}\right) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0 & 0.8 & 0.2 \\ 0 & 0.3 & 0.7 \end{bmatrix} \end{array}$$

Calculate $H(X), H(Y), H(X,Y), H({}^{X}/_{Y})$ and $H({}^{Y}/_{X})$ with this channel.

(b) (i) Let $P_X(X = 0) = P_X(X = 1) = 0.5$, $Q_X(X = 0) = 0.25$, $Q_X(X = 1) = 0.75$ *and* $R_X(X = 0) = 0.2$, $R_X(X = 1) = 0.8$. Show that triangle inequality does hold for divergence, i.e.

$$D(P_X \| R_X) > D(P_X \| Q_X) + D(Q_X \| R_X)$$

(ii) Give an example of two distributions p and q on a binary alphabet such that $D(p\|q) = D(q\|p)$ (other than the trivial case $p = q$).

6 + (4 + 2) = 12

7. (a) Calculate the mutual information between X and Y, using the following conditional probability matrix:

$$P\left(X/_Y\right) = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array}\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \end{array}$$

Given that $p(y_1) = 0.6, p(y_2) = 0.3$ and $p(y_3) = 0.1$.

(b) If f is a concave function and X is a random variable, then prove that (in general)

$$E[f\ (X)] \leq f\ (E(X)).$$

(c) Show that for independent discrete random variables X and Y, $I(X; X + Y) - I(Y; X + Y) = H(X) - H(Y)$.

5 + 4 + 3 = 12

Group – E

8. (a) Let us consider the following joint probability mass function on (X, Y)

$$P\ (X, Y) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} \frac{1}{4} & \frac{1}{6} & \frac{1}{12} \\ \frac{1}{24} & \frac{1}{8} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{24} & \frac{1}{8} \end{bmatrix} \end{array}$$

(i) Find the minimum probability of error estimator $\hat{X}(Y)$ and the associated probability.

(ii) Evaluate Fano's inequality for this problem and compare.

(b) Let

$$X = \begin{cases} 1, & \text{with probability } \frac{1}{2} \\ 2, & \text{with probability } \frac{1}{4} \\ 3, & \text{with probability } \frac{1}{4}. \end{cases}$$

Let $X_1, X_2, \cdots$ be drawn $i.i.d.$ according to this distribution. Find $lim_{n\to\infty}[P(X_1, X_2, \cdots, X_n)]^{\frac{1}{n}}$.

6 + 6 = 12

9. (a) Consider a sequence of $i.i.d.$ binary random variables, $X_1, X_2, \cdots X_n$, where the probability that $X_i = 1$ is 0.65.

(i) With $n = 20$ and $\in = 0.15$, which sequences fall in the typical set $A_{\in}^{(n)}$? What is the probability of the typical set? How many elements are there in the typical set?

(ii) How many elements are there in the smallest set that has probability 0.85?

(iii) How many elements are there in the intersection of the sets in parts (i) and (ii)? What is the probability of this intersection? (**Use Table-I**)

(b) Let the transition probability matrix of a two-state Markov chain $X \to Y \to Z$ is given as follows:

$$P = \begin{pmatrix} 0.25 & 0.75 \\ 0.6 & 0.4 \end{pmatrix}$$

Verify Data Processing lemma for the above problem. $(i.e. I(X;Z) \leq I(X;Y))$

6 + 6 = 12

Department & Section	**Submission Link**
AEIE	https://classroom.google.com/c/Mjk5MDQzODA0ODk4/a/MzQ0MDUyOTcxNjc1/details
ECE	https://classroom.google.com/c/Mjk5MDQzODA0NzYx/a/MzQ0MDUyNTUxMTQ5/details
IT	https://classroom.google.com/c/Mjk5MDQ5ODk2MTcw/a/MzQ0MDUwNjc0NTU3/details

Table-I

k	$\binom{n}{k}$	$\binom{n}{k} p^k (1-p)^{(n-k)}$	**Distribution Function**	$p(x^n) = p^k (1-p)^{(n-k)}$	$-\frac{1}{n} log_2 p(x^n)$
0	1	7.60958E-10	7.60958E-10	7.60958E-10	1.514573173
1	20	2.82642E-08	2.90251E-08	1.41321E-09	1.469918933
2	190	4.98661E-07	5.27686E-07	2.62453E-09	1.425264693
3	1140	5.5565E-06	6.08419E-06	4.87413E-09	1.380610453
4	4845	4.38567E-05	4.99409E-05	9.05195E-09	1.335956214
5	15504	0.000260634	0.000310575	1.68108E-08	1.291301974
6	38760	0.001210087	0.001520662	3.122E-08	1.246647734
7	77520	0.004494608	0.00601527	5.798E-08	1.201993494
8	125970	0.013564085	0.019579355	1.07677E-07	1.157339254
9	167960	0.033587259	0.053166614	1.99972E-07	1.112685015
10	184756	0.068613972	0.121780586	3.71376E-07	1.068030775
11	167960	0.115841771	0.237622357	6.89699E-07	1.023376535
12	125970	0.161351038	0.398973395	1.28087E-06	0.978722295
13	77520	0.184401186	0.583374582	2.37876E-06	0.934068055
14	38760	0.171229673	0.754604255	4.41769E-06	0.889413816
15	15504	0.127199186	0.881803441	8.20428E-06	0.844759576
16	4845	0.073820956	0.955624397	1.52365E-05	0.800105336
17	1140	0.032257897	0.987882293	2.82964E-05	0.755451096
18	190	0.009984587	0.99786688	5.25505E-05	0.710796856
19	20	0.001951874	0.999818755	9.75937E-05	0.666142617
20	1	0.000181245	1	0.000181245	0.621488377