Group – E

8. (a) Let G be a simple connected planar graph with n vertices, e edges and f faces, where, $n \geq 3$. Prove that

   (i) $e \geq \frac{3}{2} f$

   (ii) $e \leq 3n - 6$.

   (b) Suppose three boys b_1, b_2 and b_3 know four girls g_1, g_2, g_3 and g_4 as follows:

   $$b_1 \rightarrow \{g_2, g_3, g_4\}, \qquad b_2 \rightarrow \{g_1, g_3\}, \qquad b_3 \rightarrow \{g_1, g_2\}$$

   (i) Draw the bipartite graph corresponding to this relationship.

   (ii) Check the existence of a complete matching for this problem.

6 + 6 = 12

9. (a) (i) Draw the dual of the graph shown in Fig. 2.

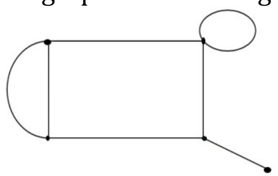

Fig. 2

   (ii) Using Kuratowski's theorem, check whether the graph shown in Fig. 3 is planar or not.

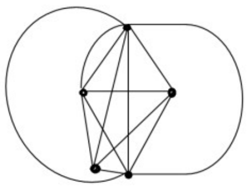

Fig. 3

   (b) Using the Decomposition Theorem, determine the chromatic polynomial of the graph shown in Fig. 4.

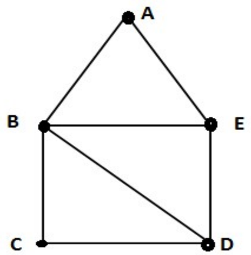

Fig. 4

(3 + 3) + 6 = 12

ADVANCED DISCRETE MATHEMATICS AND STATISTICAL METHODS (MATH 5101)

Time Allotted : 3 hrs **Full Marks : 70**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 5 (five) from Group B to E, taking at least one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A
(Multiple Choice Type Questions)

1. Choose the correct alternative for the following: **10 × 1 = 10**

   (i) A fair coin is tossed twice. The probability of getting exactly two tails is

   (a) $\frac{1}{8}$ (b) $\frac{3}{8}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$.

   (ii) If for a random variable X, $Var(X) = 1$, then $Var(2X + 3)$ is

   (a) 1 (b) 2 (c) 4 (d) 5.

   (iii) The moment generating function of binomial distribution $B(n, p)$ where, $q = (1 - p)$ is

   (a) $(p + qe^t)^n$ (b) $(pe^t + q)^n$ (c) $(p + qe^t)$ (d) $(pe^t + q)$.

   (iv) If X is normally distributed random variable with zero mean and unit variance, then the expectation of X^2 is

   (a) 1 (b) 2 (c) 8 (d) 0.

   (v) If $u + 3x = 5, 2y - v = 7$ and the correlation coefficient of x and y is 0.12, then the correlation coefficient of u and v is

   (a) 1 (b) 0 (c) 0.12 (d) -0.12.

   (vi) The sum of the coefficients in the expansion of $(w + x + y + z)^5$ is

   (a) 2^5 (b) 3^5 (c) 4^5 (d) 5^5.

   (vii) The generating function of the sequence $\{1, 1, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \ldots\}$ is

   (a) e^x (b) e^{-x} (c) $\log(1+x)$ (d) $(1-x)^{-1}$.

   (viii) A connected planar graph with 5 edges determines 3 regions. The number of vertices of this graph is

   (a) 3 (b) 4 (c) 5 (d) 6.

(ix) If a simple connected graph has at least one edge, then the sum of the coefficients in its chromatic polynomial is
(a) 1 (b) 2 (c) 3 (d) 0

(x) The clique number of the graph in Fig. 1 is

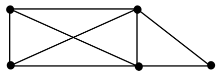

Fig. 1

(a) 2 (b) 3 (c) 4 (d) 5.

Group – B

2. (a) Let X, Y, Z be three coins in a box. Suppose X is a fair coin, Y is a two-headed coin and Z is weighted so that the probability of heads is $\frac{1}{3}$. A coin is selected at random and is tossed. If a head appears, find the probability that it is the fair coin X.

(b) The probability density function of a random variable x is given by

$$f(x)=\begin{cases} kx^2, 0 \le x \le 6 \\ k(12-x)^2, 6 \le x \le 12 \\ 0, \textit{elsewhere} \end{cases}$$

(i) Evaluate k.
(ii) Find the distribution function of X and evaluate P(4 < X < 8).

6 + 6 = 12

3. (a) Three identical boxes I, II and III contain respectively 4 white and 3 red balls, 3 white and 7 red balls, 2 white and 3 red balls. A box is chosen at random and a ball is drawn out of it. If the ball is found to be white, what is the probability that box II was selected?

(b) Let X be a random variable and a, b be real constants. Prove that
(i) $Var(aX+b) = a^2 Var(X)$
(ii) $Var(X) = E\{X(X-1)\} - m(m-1)$, where, $m = E(X)$.

6 + 6 = 12

Group – C

4. (a) The marks obtained by 1000 students in a final examination are found to be approximately normally distributed with mean 70 and standard deviation 5. Estimate the number of students whose marks are between 60 and 75 both inclusive, given that the area under the normal curve $\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$ between $z = 0$ and $z = 2$ is 0.4772 and that between $z = 0$ and $z = 1$ is 0.3413.

(b) Find the moment generating function of the Poisson distribution

$$P(X=r)=\begin{cases} \dfrac{e^{-\lambda}\lambda^r}{r!}, & r = 0, 1, 2, \ldots \\ 0, \textit{elsewhere} \end{cases}$$

Then use the moment generating function to find the mean and the variance.

6 + 6 = 12

5. (a) A random variable follows a binomial distribution with mean 4 and standard deviation $\sqrt{2}$. Find the probability that the variable assumes a non-zero value.

(b) Let us consider the bivariate data given by (x, y) as

x	-6	-4	-3	-1	1	2	4	7
y	-4	-3	-1	-1	0	2	3	4

(i) Find the correlation co-efficient of x and y.
(ii) Find the regression co-efficient of y on x.
(iii) Find the regression line of y on x.

6 + 6 = 12

Group – D

6. (a) In how many ways can 20 students out of a class of 30 be selected for an extra-curricular activity, if
(i) Rama refuses to be selected?
(ii) Raja insists on being selected?
(iii) Gopal and Govind insist on being selected?
(iv) either Gopal or Govind or both get selected?
(v) just one of Gopal and Govind gets selected?
(vi) Rama and Raja refuse to be selected together?

(b) If m is an odd positive integer, prove that there exists a positive integer n such that m divides $(2^n - 1)$.

7 + 5 = 12

7. (a) Solve the recurrence relation $a_{n+2} - 6a_{n+1} + 9a_n = 3(2^n) + 7(3^n), n \geq 0$ given that $a_0 = 1$ and $a_1 = 4$.

(b) If $(n+1)$ integers not exceeding $2n$ are selected, show that there must be an integer that divides one of the other integers. Deduce that if 151 integers are selected from $\{1, 2, 3, \ldots, 300\}$ then the selection must include two integers x and y, one of which divides the other.

7 + 5 = 12