(b) Let V and W be vector spaces over a field F. $T: V \rightarrow W$ be a linear mapping. Then prove that $Ker\, T = \{\alpha \in V : T(\alpha) = \theta'\}$, where, θ' is the zero vector in W, is a subspace of V.

6 + 6 = 12

9. (a) Let $T:\mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $S:\mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $T(x,y) = (2x+y, 0), (x,y) \in \mathbb{R}^2$ and $S(x,y) = (x+y, xy), (x,y) \in \mathbb{R}^2$ respectively. Determine whether $S \circ T$ is a linear transformation.

(b) Determine the linear mapping $T:\mathbb{R}^3 \rightarrow \mathbb{R}^2$ which maps the basis vectors $\{(1,0,0), (0,1,0), (0,0,1)\}$ of $\mathbb{R}^3$ to the vectors $\{(1,1), (2,3), (3,2)\}$ of $\mathbb{R}^2$ respectively. Verify Rank-Nullity theorem.

6 + 6 = 12

LINEAR ALGEBRA (MATH 4182)

Time Allotted : 3 hrs **Full Marks : 70**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and <u>any 5 (five)</u> from Group B to E, taking <u>at least one</u> from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A
(Multiple Choice Type Questions)

1. Choose the correct alternative for the following: **10 × 1 = 10**

(i) The product of the eigen values of a square matrix A is
(a) traceA (b) det A (c) 0 (d) 1.

(ii) The geometric multiplicity of the eigenvalue 0 of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ is
(a) 0 (b) 1 (c) 2 (d) 3.

(iii) If the nullity of the matrix $\begin{pmatrix} k & 1 & 2 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{pmatrix}$ is 1, then the value of k is
(a) −1 (b) 0 (c) 1 (d) 2.

(iv) The linear map $T:\mathbb{R}^4 \rightarrow \mathbb{R}^4$ of the matrix $M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ with respect to the standard basis is
(a) $T(x,y,z,w) = (z,y,x,w)$ (b) $T(x,y,z,w) = (0,x,w,y)$
(c) $T(x,y,z,w) = (z,x,w,y)$ (d) $T(x,y,z,w) = (x-y,0,y,w)$.

(v) Let $T: P_3[0,1] \rightarrow P_2[0,1]$ be defined by $(Tp)(x) = p''(x) + p'(x)$. Then the matrix representation of T with respect to the bases $\{1, x, x^2, x^3\}$ and $\{1, x, x^2\}$ of $P_3[0,1]$ and $P_2[0,1]$ respectively is
(a) $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 6 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 2 & 1 & 0 \\ 6 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 2 \\ 3 & 6 & 0 \end{pmatrix}$.

(vi) Consider the following vectors in $\mathbb{R}^3$: $u_1 = (1,2,1), u_2 = (2,1,-4), u_3 = (3,-2,1)$. Which one of the following statement is true?
(a) u_1 is parallel to u_2
(b) u_2 is parallel to u_3
(c) all of them are orthogonal to each other
(d) only u_1 and u_2 are orthogonal to each other but u_3 is not.

(vii) The dimension of the subspace $\{(x_1,x_2,x_3,x_4,x_5):3x_1-x_2+x_3=0\}$ of $\mathbb{R}^5$ is
(a) 1 (b) 2 (c) 3 (d) 4.

(viii) The dimension of a vector space is
(a) the number of vectors in a basis
(b) the number of subspaces
(c) the dimension of its subspaces
(d) the number of vectors in a spanning subset.

(ix) For a diagonalisable square matrix which one of the followings is not true?
(a) all its eigen vectors are linearly independent
(b) all its eigen values are regular
(c) all its eigen values are distinct
(d) it is similar to a diagonal matrix.

(x) The real quadratic form in two variables a, b associated with the matrix $\begin{pmatrix}5 & 1\\ 1 & -1\end{pmatrix}$ is
(a) $5a^2 + 2ab - b^2$ (b) $5a^2 - 2ab + b^2$
(c) $a^2 + 5ab - b^2$ (d) $a^2 - 5ab + b^2$.

Group – B

2. (a) Find an orthogonal matrix P such that $D = P^{-1}BP$ is diagonal, where, $B = \begin{pmatrix}1 & -3 & 3\\ 3 & -5 & 3\\ 6 & -6 & 4\end{pmatrix}$.

(b) Reduce the quadratic form $Q(x,y,z) = x^2+2y^2+3z^2-2xy+4yz$ to the normal form and show that it is indefinite. Find the rank and signature of Q.

6 + 6 = 12

3. (a) Find the Singular Value Decomposition of $A = \begin{pmatrix}3 & 2 & 2\\ 2 & 3 & -2\end{pmatrix}$.

(b) Find the generalized inverse G of the matrix $A = \begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}$.

8 + 4 = 12

Group – C

4. (a) Let U and W be two subspaces of a vector space V over a field F. Then prove that U+W is a subspace of V, where, $U+W=\{u+w : u\in U, w\in W\}$ is the linear sum of the subspaces U and W.

(b) Examine whether the set S is a subspace of the vector space $\mathbb{R}_{2\times2}$ where, S is the set of all 2×2 real diagonal matrices.

6 + 6 = 12

5. (a) Find for what values of h the following set of vectors is linearly independent.

$$\left\{v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, v_2 = \begin{bmatrix}h\\1\\-h\end{bmatrix}, v_3 = \begin{bmatrix}1\\2h\\3h+1\end{bmatrix}\right\}$$

(b) Find a basis and the dimension of the subspaces S, T and $S\cap T$ of $\mathbb{R}^3$ where $S = \{(x,y,z) \in \mathbb{R}^3 : x + 2y - z = 0\}$ and $T = \{(x,y,z) \in \mathbb{R}^4 : 2x - y + 3z = 0\}$. Also find the dimension of the subspace $S + T$ of $\mathbb{R}^3$.

5 + 7 = 12

Group – D

6. (a) Use Gram-Schmidt process to obtain an orthonormal basis of the subspace of the Euclidean space $\mathbb{R}^4$ with standard inner product, generated by the linearly independent set $\{(1,1,0,1), (1,1,0,0), (0,1,0,1)\}$.

(b) Consider $f(t)=3t-5, g(t)=t^2$ in the polynomial space $P(t)$ with inner product $\langle f,g\rangle = \int_0^1 f(t)g(t)dt$. Find (i) $\langle f,g\rangle$ and (ii) $\|f\|, \|g\|$.

6 + 6 = 12

7. (a) Prove that an orthogonal set of nonzero vectors in a Euclidean space V is linearly independent.

(b) Let S consist of the following vectors in $\mathbb{R}^4$: $u_1 = (1,1,0,-1), u_2 = (1,2,1,3), u_3 = (1,1,-9,2), u_4 = (16,-13,1,3)$. Show that S is orthogonal and a basis of $\mathbb{R}^4$.

6 + 6 = 12

Group – E

8. (a) Let V be a vector space of n-square real matrices and M be an arbitrary but fixed matrix in V. $T: V \to V$ be defined by $T(A) = AM + MA$, where, A is any matrix in V. Examine if T is linear.