B.TECH/IT/4TH SEM/MATH 2203/2019

GRAPH THEORY AND ALGEBRAIC STRUCTURES
(MATH 2203)

Time Allotted : 3 hrs **Full Marks : 70**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 5 (five) from Group B to E, taking at least one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A
(Multiple Choice Type Questions)

1. Choose the correct alternative for the following: **10 × 1 = 10**

(i) Which of the following operations is not commutative?
(a) Matrix addition (b) Arithmetical multiplication
(c) Matrix multiplications (d) Arithmetical addition.

(ii) Let G be a simple connected planar graph with 13 vertices and 19 edges. Then, the number of faces in the planar embedding of the graph is
(a) 6 (b) 8 (c) 9 (d) 13.

(iii) Let G be a group and a∈G. If O(a)=17 then O(a^8) is
(a) 17 (b) 16 (c) 8 (d) 5.

(iv) The ring of matrices $\left\{\begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix} : a, b \in \mathbb{Z}\right\}$ with respect to usual matrix operations
(a) contains divisors of zero
(b) does not contain divisors of zero
(c) contains unity
(d) is a field.

(v) The symmetric group S_3 is
(a) cyclic but not abelian (b) cyclic and abelian
(c) non cyclic and non abelian (d) none.

(vi) A tree with 5 vertices can be coloured by 4 colours in ________ number of ways.
(a) 324 (b) 350 (c) 20 (d) 120.

(vii) If x is an element of a group G and $O(x) = 5$, then
(a) $O(x^{10}) = 5$ (b) $O(x^{15}) = 5$
(c) $O(x^{23}) = 5$ (d) $O(x^{20}) = 5$.

(viii) If H is a subgroup of a group G and a, b are two distinct elements of G, then indicate which of the following statements is true:
(a) $aH = Ha$ (b) $Ha \cap Hb = \phi$
(c) $Ha \cap Hb \neq \phi$ and $Ha \neq Hb$ (d) $aH = bH$.

(ix) Which one of the following rings is an integral domain?
(a) $\mathbb{Z}_{101}$ (b) $\mathbb{Z}_{200}$ (c) $\mathbb{Z}_{2001}$ (d) $\mathbb{Z}_{356}$.

(x) The number of elements of order 6 in the cyclic group of order 42 is
(a) 7 (b) 2 (c) 3 (d) 42.

Group – B

2. (a) Prove that $\lambda^4 - 3\lambda^3 + 4\lambda^2$ cannot be a chromatic polynomial of a graph G.

(b) Define dual of a graph. Draw the dual of the graph K_4.

(c) Is K_5 planar? Justify your answer.

4 + 5 + 3 = 12

3. (a) (i) State Hall's Marriage Theorem.
(ii) Find the number of perfect matchings in K_n.

(b) How many edges a planar graph must have with 5 regions and 7 vertices? Draw one such graph.

(2 + 6) + 4 = 12

Group – C

4. (a) Show that all roots of the equation $x^4 = 1$ forms a commutative group under the usual multiplication.

(b) Prove that in a group G for all $a, b \in G$, the equation $ax = b$ has a unique solution in G.

(c) Show that a group $(G,*)$ is abelian if and only if $(a * b)^2 = a^2 * b^2 \; \forall a, b \in G$.

4 + 4 + 4 = 12

5. (a) Prove that a group is abelian if and only if $(ab)^{-1}=a^{-1}b^{-1}$.

(b) Let A=$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$ and B=$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix}$ be two permutations. Show that AB≠BA.

(c) Find the order of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$.

6 + 4 + 2 = 12

Group – D

6. (a) Prove that a group $(G,*)$ is commutative if $(a * b)^n = a^n * b^n$, for any three consecutive integers n and for all $a, b \in G$.

(b) Prove that every group of order less than 6 is abelian.

6 + 6 = 12

7. (a) Prove that every group of prime order is cyclic.

(b) Prove that every cyclic group is abelian. Is the converse true? Justify your answer.

6 + (4 + 2) = 12

Group – E

8. (a) Let $G = (\mathbb{R}, +)$, $G' = (\{z \in \mathbb{C}: |z| = 1\}, \cdot)$ and $\phi: G \to G'$ is defined by $\phi(x) = \cos 2\pi x + i \sin 2\pi x,\ x \in \mathbb{R}$. Determine whether ϕ is a homomorphism.

(b) Let R be a ring. The centre of R is the subset $Z(R)$ defined by $Z(R) = \{x \in R: xr = rx \ \forall r \in R\}$. Prove that $Z(R)$ is a subring of R.

6 + 6 = 12

9. (a) Prove that every finite integral domain is a field.

(b) Define divisors of zero in a ring R. Show that the matrix $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is a divisor of zero in $M_2(\mathbb{Z})$.

(c) Show that a division ring contains exactly two idempotent elements.

6 + 3 + 3 = 12