B.TECH/CSE/ECE/8TH SEM/MATH 4282/2019

# ADVANCED COMPUTATIONAL MATHEMATICS AND GRAPH THEORY (MATH 4282)

**Time Allotted : 3 hrs** **Full Marks : 70**

***Figures out of the right margin indicate full marks.***

***Candidates are required to answer Group A and any 5 (five) from Group B to E, taking at least one from each group.***

***Candidates are required to give answer in their own words as far as practicable.***

## Group – A
## (Multiple Choice Type Questions)

1. Choose the correct alternative for the following: **10 × 1 = 10**

(i) The Eulerian number $\left\langle {4 \atop 3} \right\rangle$ is
(a) 1 (b) 3 (c) 4 (d) 11.

(ii) The Stirling number of the first kind (or Stirling cycle number) $\left[{4 \atop 1}\right]$ is
(a) 24 (b) 6 (c)4 (d) 1.

(iii) The tenth Bernoulli number $B_{10} =$
(a) $-\frac{1}{30}$ (b) $\frac{1}{42}$ (c) $\frac{5}{66}$ (d) $-\frac{691}{2730}$

(iv) The generating function of the sequence $\{1, 2, 4, 8, 16, 32, \dots, 2^n, \dots\}$ is
(a) $\frac{1}{1+z^2}$ (b) $\frac{1}{1-z^2}$ (c) $\frac{1}{1-2z}$ (d) $\frac{1}{1+2z}$.

(v) The remainder in the division of $18! \times 3^{18}$ by 19 is
(a) 18 (b) 1 (c) 17 (d) 3

(vi) $\Delta(x^{\underline{m}}) =$
(a) $(m-1)x^{\underline{m}}$ (b) $mx^{\underline{m}}$ (c) $(m-1)x^{\underline{m-1}}$ (d) $mx^{\underline{m-1}}$

(vii) If $u_0 = 2$ , $u_n = 3u_{n-1}$ for $n > 0$, then $u_{100} + 7$ is
(a) even
(b) divisible by 3
(c) divisible by 7
(d) neither divisible by 3 nor divisible by 7.

(viii) Let $[x]$ denote the greatest integer less than or equal to $x$. If $c > 0$ and $[\sqrt[100]{c}] = 0$, then
(a) $2 < c < 3$ (b) $1 < c < 2$
(c) $100 < c < 200$ (d) $c < 1$





(ix) The chromatic polynomial of a complete graph with $n = 3$ vertices is
(a) $\lambda(\lambda-1)$ (b) $\lambda(\lambda-1)(\lambda-2)$ (c) $\lambda^3$ (d) $(\lambda-1)$

(x) A graph $G$ has a spanning tree if and only if $G$ is
(a) disconnected (b) a tree
(c) complete (d) connected.

## Group – B

2. (a) Prove the following recurrence relation for $\left\{{n \atop k}\right\}$, the Stirling number of the second kind (or Stirling subset number):
$$\left\{{n \atop k}\right\} = k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\}.$$
Given that $\left\{{1 \atop 0}\right\} = 0, \left\{{1 \atop 1}\right\} = 1, \left\{{1 \atop 2}\right\} = 0, \left\{{2 \atop 0}\right\} = 0, \left\{{2 \atop 3}\right\} = 0,$ calculate $\left\{{3 \atop 1}\right\}, \left\{{3 \atop 2}\right\}, \left\{{3 \atop 3}\right\}$ by using the recurrence relation.

(b) Let $F_n$ denote the $n$th Fibonacci number and let the generating function of the Fibonacci numbers be
$$F(z) = F_0 + F_1 z + F_2 z^2 + F_1 z^3 + F_1 z^4 + \cdots = \sum_{n \geq 0} F_n z^n$$
Prove that $F(z) = \frac{z}{1-z-z^2}$. Show your calculations in detail.

**7 + 5 = 12**

3. (a) (i) State the recurrence relation for $J(n)$ in the Josephus problem.
(ii) Calculate $J(n)$ upto $n = 32$ and show the values in a table.
(iii) Indicating the pattern that you find in the data, state the solution of the problem (i.e., an expression for $J(n)$). (The proof is not required).

(b) Recall the generating function of the Bernoulli numbers, $B_n$:
$\frac{z}{e^z-1} = \sum_{n\geq 0} B_n \frac{z^n}{n!}$.
Prove that $z \cot z = \sum_{n\geq 0} (-4)^n B_{2n} \frac{z^{2n}}{(2n)!}$.

**(2 + 2 + 2) + 6 = 12**

## Group – C

4. (a) Compute $\varphi(999)$ where $\varphi(n)$ is the Euler $\varphi$-function. Show your calculations in detail and state any result that you use.

(b) State Fermat's little theorem. Use this theorem to find the remainder in the division of $2^{205} + 3^{304} + 4^{402}$ by 11. Show your work in detail.

**5 + 7 = 12**

5. (a) Let $[x]$ denote the greatest integer less than or equal to $x$. Prove that $\left[\sqrt{[x]}\right] = \left[\sqrt{x}\,\right]$, for real $x \geq 0$.

(b) Prove that $\sum_k \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$, for integer $n$.

**6 + 6 = 12**

## Group – D

6. (a) Find the greatest common divisor of the Fibonacci numbers $F_9 = 34$ and $F_{10} = 55$ by using the Euclidean algorithm. Express it as $55x + 34y$ where $x$ and $y$ are integers. Show your calculations in detail.

(b) Show that $\sum_{k=0}^{m} \binom{m}{k} / \binom{n}{k} = \frac{n+1}{n+1-m}$.

**6 + 6 = 12**

7. (a) Let the harmonic numbers be defined as $H_n \coloneqq \sum_{k=1}^{n} \frac{1}{k}$ and the harmonic numbers of order $r$ be defined as $H_n^{(r)} \coloneqq \sum_{k=1}^{n} \frac{1}{k^r}$ (Note that $H_n^{(1)} = H_n$). Show that $H_n - \ln n = 1 - \frac{1}{2}\left(H_n^{(2)} - 1\right) - \frac{1}{3}\left(H_n^{(3)} - 1\right) - \frac{1}{4}\left(H_n^{(4)} - 1\right) - \ldots$

(b) Let $T_n$ denote the minimum number of moves that will transfer $n$ disks from one peg to another in the Tower of Hanoi problem. Show that

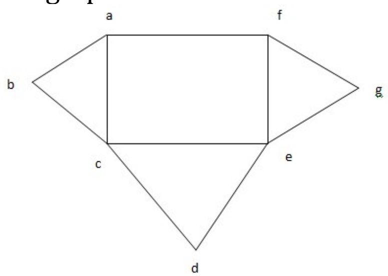
$$T_3 \leq 2T_2 + 1.$$

Prove the recursion $T_n \leq 2T_{n-1} + 1$, for $n > 0$, without using induction. State clearly why your proof is valid only for "≤" in the recursion and not for "=".

**6 + 6 = 12**

## Group – E

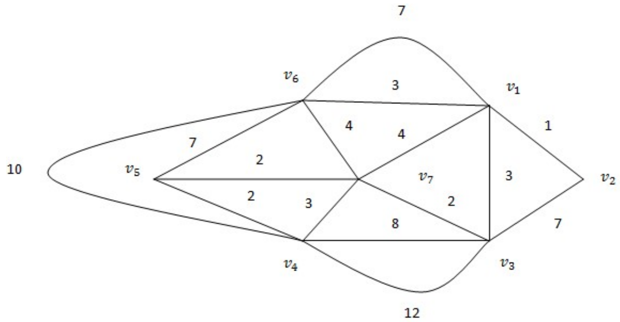
8. (a) Find all the maximal and maximum matchings of the graph given below. Find the matching number of the graph. Check if there exists a perfect matching in the graph.





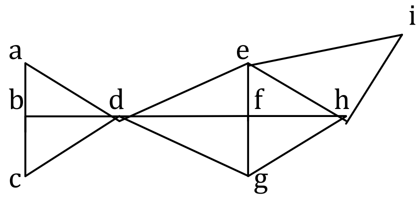



(b) Apply Dijkstra's Algorithm to find the shortest path from $v_2$ to $v_5$ in the given graph:



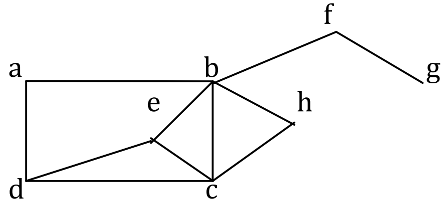

**5 + 7 = 12**

9. (a)




Find the vertex connectivity and the edge connectivity of the given graph.

(b)




Find all the cut sets in the given graph.

(c) Can $\lambda^4 - 4\lambda^3 + 7\lambda^2 - 2\lambda + 3$ be the chromatic polynomial of a simple graph having at least one edge? Justify your answer.

**4 + 4 + 4 = 12**