NUMBER THEORY & ALGEBRAIC STRUCTURES
(MATH 2201)

Time Allotted : 3 hrs **Full Marks : 70**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 5 (five) from Group B to E, taking at least one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A
(Multiple Choice Type Questions)

1. Choose the correct alternative for the following: **10 × 1 = 10**

(i) If the cyclic group G contains 12 distinct elements then the number of possible generators of the group is
(a) 1 (b) 2 (c) 3 (d) 4.

(ii) The linear congruence $5x \equiv 3 (\mathrm{mod}\ 11)$ has
(a) unique solution modulo 11 (b) no solutions
(c) 10 incongruent solutions (d) 11 incongruent solutions.

(iii) Consider the lattice $(S, \leq)$, where $S = \{1,2,4,5,8,9\}$ and $' \leq '$ denotes usual "less than or equal to" relation. Then
(a) $4 \wedge (5 \vee 9) = 2 \vee (2 \wedge 8)$
(b) $4 \wedge (5 \vee 9) = 2 \vee (2 \wedge 5)$
(c) $4 \wedge (5 \vee 9) = (2 \vee (2 \wedge 8)) \vee 4$
(d) $4 \wedge (5 \vee 9) = (2 \vee (2 \wedge 8)) \vee 5$.

(iv) The number of generators of an infinite cyclic group is
(a) 1 (b) 2 (c) 0 (d) infinite.

(v) The order of [6] in $\mathbb{Z}_{14}$ is
(a) 2 (b) 14 (c) 7 (d) 5.

(vi) If gcd(a,b)=d then gcd(a/d, b/d) is
(a) d (b) 0 (c) 1 (d) a/d.

(vii) A group contains 12 elements. Then the possible number of elements in a subgroup is
(a) 3 (b) 5 (c) 7 (d) 11.

(viii) The number of solutions $x^2-[4]x+[3]=[0]$ in $\mathbb{Z}_{12}$ is
(a) 2 (b) 4 (c) 6 (d) 8.

(ix) Let $R = \{a + \sqrt{2}b : a, b \in \mathbb{Q}\}$. Then R is a field with respect to usual addition and multiplication. If an element of R, $a + \sqrt{2}b \neq 0$, then its multiplicative inverse is given by
(a) $\frac{1}{(a+\sqrt{2}b)^2}$ (b) $\frac{1}{a-\sqrt{2}b}$
(c) $\frac{a}{a^2-2b^2} + \sqrt{2}\frac{(-b)}{a^2-2b^2}$ (d) $\frac{a}{a^2-2b^2} + \sqrt{2}\frac{b}{a^2-2b^2}$.

(x) A zero divisor in $\mathbb{Z}_{12}$ under addition and multiplication modulo 12 is
(a) [2] (b) [5] (c) [7] (d) [11].

Group – B

2. (a) Find the remainder when $2^{1000000}$ is divided by 17.
(b) Show that 32 divides $(a^2+3)(a^2+7)$, where a is an odd integer.
(c) Define a totally ordered set with an example.
5 + 5 + 2 = 12

3. (a) Find integers x, y, z satisfying $\gcd(198,288,512) = 198x + 288y + 512z$.
(b) Solve the congruence $72x \equiv 18 (\mathrm{mod}\ 42)$. Mention the incongruent solutions modulo 42.
6 + 6 = 12

Group – C

4. (a) Give examples of binary operations which are not
(i) associative (ii) commutative.
(b) Write down the multiplicative Cayley table for U(8).
(c) Let G be group. If a, b$\in G$, then prove that $(ab)^{-1}=b^{-1}a^{-1}$.
(d) Give two examples of finite groups of order at least 4 and write down the corresponding Cayley tables.
2 + 3 + 3 + 4 = 12

5. (a) Prove that if n is a positive integer such that $n \geq 3$, then the symmetric group S_n is a noncommutative group.
(b) Let two permutations be given by $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 4 & 7 & 5 & 2 & 3 & 1 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 6 & 7 & 3 & 5 & 2 \end{pmatrix}$
(i) Write α as a product of disjoint cycles.

(ii) Write β as a product of $2-$cycles.
(iii) Is β an even permutation?
(iv) Is α^{-1} an even permutation?

4 + (2 + 2 + 2 + 2) = 12

Group – D

6. (a) Let G be a group and a be an element of G such that $O(a) = n$. Prove that
(i) If $a^m = e$ for some positive integer m, then n divides m.
(ii) For every positive integer t, $O(a^t) = \frac{n}{\gcd(t,n)}$

(b) Let H be a subgroup of a group G. Show that for any $g \in G$, $K = gHg^{-1} = \{ghg^{-1}:\ h \in H\}$ is a subgroup of G.

(3 + 4) + 5 = 12

7. (a) Let G $= \langle a \rangle$ be a cyclic group of order n. Then for any integer r, $1 \leq r < n$, a^r is a generator of G iff gcd(r,n)=1.

(b) Prove that any two right or left cosets of a subgroup are either disjoint or identical.

6 + 6 = 12

Group – E

8. (a) Determine whether the given map ϕ is a homomorphism:
$\phi : G \to G_1$ where $G = S_3, G_1 = \{1, -1\}$ and
$$\left.\begin{aligned}\phi(\sigma) &= 1 \quad if\ \sigma\ is\ even\\ &= -1\ if\ \sigma\ is\ odd\end{aligned}\right\}$$

(b) Find the units in the ring $\mathbb{Z}_8$.

(c) Let R be a commutative ring with characteristic p, p is prime. Prove that $(a+b)^p = a^p + b^p$, $a, b \in R$.

5 + 2 + 5 = 12

9. (a) Give an example of a finite ring R with unity and a subring S of R such that S contains no unity. Provide appropriate justifications.

(b) Prove that the ring $(\mathbb{Z}_n, +, \cdot)$ is an integral domain if and only if n is prime.

6 + 6 = 12