ADVANCED DISCRETE MATHEMATICS AND STATISTICAL METHODS
(MATH 5101)

Time Allotted : 3 hrs **Full Marks : 70**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 5 (five) from Group B to E, taking at least one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A
(Multiple Choice Type Questions)

1. Choose the correct alternative for the following: **10 × 1 = 10**

(i) Suppose the probability that an item produced by a certain machine will be defective is 0.1. The probability that a sample of 10 items will contain at most one defective item is (assuming that the quality of successive items is independent)
(a) 0.1123 (b) 0.3423 (c) 0.7361 (d) 1

(ii) If S is the set containing all permutations of the numbers 1, 2, 3 and we choose randomly a permutation from the set, then the probability that it is a derangement is
(a) 0.33 (b) 0 (c) 1 (d) 0.5.

(iii) The sequence corresponding to the generating function $(3+x)^3$ is
(a) { 9,18,18,1,0,0,0,....} (b) {0,0,0,27,27,9,.....}
(c) {27,27,9,1,0,0,.....} (d) {0,0,0,0,0,0,0,.....}.

(iv) A and B are two events such that $B \subseteq A$. Then,
(a) $P(B) \le P(A)$ (b) $A \cap B = \varphi$
(c) $P(B) > P(A)$ (d) $P(A \cap B) = P(A)$

(v) If $\varphi(t) = E\left[e^{tX}\right]$ is the moment generating function of a random variable X, then $\varphi'(0) =$
(a) $\mathrm{Var}(X)$ (b) $E(X)$ (c) $E(X^2)$ (d) $E(X^3)$

(vi) The number of internal vertices in a binary tree with n vertices is
(a) $\frac{n-1}{2}$ (b) $\frac{n+1}{2}$ (c) $\frac{n}{2}$ (d) $\frac{n}{4}$

(vii) If G is a 5-vertex colourable graph but not 4-vertex colourable graph, then
(a) $\chi(G) = 4$ (b) $\chi(G) = 5$ (c) $\chi(G) = 9$ (d) $\chi(G)$ does not exist.

(viii) Total number of non-negative integral solution to the equation $x_1 + x_2 = 10, x_1, x_2 \ge 0$ is
(a) 10 (b) 11 (c) 9 (d) 12

(ix) The number of ways 5 boys and 4 girls can sit at a round table if there is no restriction is
(a) 5! (b) 4! (c) 8! (d) 9!

(x) A connected planar graph with 5 vertices determines 3 regions. The number of edges of the graph is
(a) 6 (b) 5 (c) 2 (d) 3

Group – B

2. (a) An online computer system has four incoming communication lines with the properties described in the table below:

Line	Fraction of traffic	Fraction of messages without error
1	0.4	0.9998
2	0.3	0.9999
3	0.1	0.9997
4	0.2	0.9996

(i) What is the probability that a randomly chosen message is received without errors?
(ii) If a message is received without errors, what is the probability that it came from line 3?

(b) In answering a question on a multiple-choice test, a student either knows the answer or guesses. Let p be the probability that she knows the answer and $(1-p)$ the probability that she guesses. Assume that a student who guesses the answer will be correct with probability $1/m$, where m is the number of multiple-choice alternatives. What is the conditional probability that a student actually knew the answer to a question given that she answered it correctly?

6 + 6 = 12

3. (a) If the random variable X takes the values 1, 2, 3 and 4 such that $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$, find the probability mass function and probability distribution function of X.

(b) Suppose that X is a continuous random variable whose probability density function is given by:

$$f(x)=\begin{cases} c(4x-2x^2), & 0<x<2 \\ 0, & \text{otherwise} \end{cases}$$

(i) Compute the value of c.

(ii) Find $P(X>1)$.

(iii) Find the distribution function for X.

6 + (2 + 2 + 2) = 12

Group – C

4. (a) If the weekly wage of 10,000 workers in a factory follows normal distribution with mean and standard deviation Rs. 70 and Rs. 5 respectively , find the number of workers whose weekly wages are

(i) between Rs.66 and Rs.72

(ii) less than Rs.66

(iii) more than Rs.72.

(b) For two variables x and y the equations of two regression lines are $x+4y+3=0$ and $4x+9y+5=0$. Identify which one is of "x on y". Find the correlation coefficient.

6 + 6 = 12

5. (a) Prove that the moment generating function of the sum of independent random variables is just the product of the individual moment generating functions.

(b) Find the moment generating function of the exponential distribution:

$$f(x)=\begin{cases} \lambda e^{-\lambda x}, & x>0 \\ 0 & \text{otherwise} \end{cases}$$

Use the function to find the mean and variance of the distribution.

6 + 6 = 12

Group – D

6. (a) How many solutions does the equation $x_1+x_2+x_3=11$ have, where $x_1+x_2+x_3=11$ are non-negative variables such that $x_1\le 3, x_2\le 4, x_3\le 6$? Use the principle of inclusion-exclusion.

(b) How many positive integers n can be formed using the digits $3, 4, 4, 5, 5, 6, 7$ if n has to exceed 5000000?

6 + 6 = 12

7. (a) Use the method of generating function to solve the recurrence relation: $a_{n+1}-8a_n+16a_{n-1}=4^n,\ n\ge 1,\ a_0=1, a_1=8$

(b) There are 3 piles of identical red, blue and green balls where each pile contains at least 10 balls. In how many ways can 10 balls be selected

(i) if there is no restriction?

(ii) if at least one red ball must be selected?

(iii) if at least one red ball, at least two blue balls and at least three green balls must be selected?

(iv) if exactly one red ball must be selected?

(v) if exactly one red ball and at least one blue ball must be selected?

(vi) if at least one red ball is selected?

6 + 6 = 12

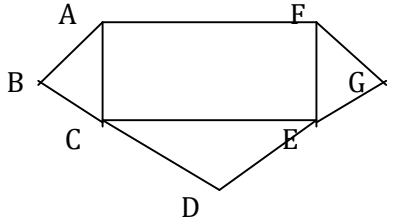
Group – E

8. (a) Prove that a planar graph G with n vertices, e edges and k components determines $f=e-n+k+1$ number of regions.

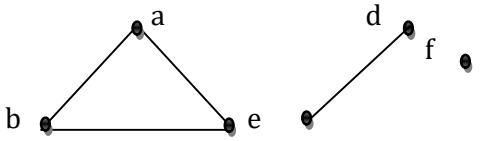
(b) Applicants a_1, a_2, a_3, a_4 apply for 5 posts p_1, p_2, p_3, p_4, p_5. The applications are done as $a_1\to\{p_1,p_2\}, a_2\to\{p_1,p_3,p_5\}, a_3\to\{p_1,p_2,p_3,p_5\}, a_4\to\{p_3,p_4\}$. Using techniques from graph theory, find whether there is any perfect matching of the set of applicants into the set of posts.

2 + 4 + 6 = 12

9. (a) Find a maximal matching, maximum matching and matching numbers of the following graph. Define a perfect matching and find whether there exists a perfect matching in the graph.



(b) Find the chromatic polynomial and hence the chromatic number for the given graph having vertices a to f



6 + 6 = 12