GRAPH ALGORITHMS
(CSEN 4163)

Time Allotted : 3 hrs **Full Marks : 70**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 5 (five) from Group B to E, taking at least one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A
(Multiple Choice Type Questions)

1. Choose the correct alternative for the following: **10 × 1 = 10**

(i) Which of the following problems is polynomial-time solvable?
(a) Longest path problem
(b) Finding a Hamiltonian circuit in a given graph
(c) Finding an Eulerian circuit in a given graph
(d) None of the above.

(ii) In a flow network, there exists a cut of value 11. Which of the following can be concluded?
(a) The minimum flow in the network is 11
(b) The maximum flow in the network ≤ 11
(c) The maximum flow in the network ≥ 11
(d) None of the above.

(iii) Time complexity to identify strongly connected component (SCC) in a graph G = (V, E) is
(a) Θ(V+E) (b) Θ(V.E) (c) Θ(E) (d) none of the above.

(iv) The number of perfect matchings in K_{2n} is
(a) (2n -1)(2n – 3)…..(3)(1) (b) 2n! / ((n!)(n!))
(c) 2n! / ((n!)(2^n)) (d) both (a) and (c).

(v) Which of the following can be a valid degree sequence for a simple undirected graph with |V| = 10?
(a) 9, 9, 9, 9, 9, 9, 9, 9, 9, 9 (b) 9, 9, 8, 7, 7, 6, 6, 5, 5, 1
(c) 9, 9, 9, 5, 4, 4, 4, 3, 3, 2 (d) none of the above.

(vi) Time complexity of Hopcroft-Karp algorithm for computing the maxium cardinality matching of a bipartite graph G = (V, E) is
(a) O(E + V) (b) O(E.V)
(c) O(E √V) (d) none of the above.

(vii) Perfect elimination ordering
(a) is an ordering of the vertices of the graph such that, for each vertex v, v and the neighbours of v that occur after v in the order form a clique
(b) is an order of the vertices we get from topological sort
(c) the order of the vertices in a perfect graph such that degree of vertices are in ascending order
(d) none of the above.

(viii) Colouring of a graph with minimum number of colours is
(a) NP-hard but not NP-complete (b) NP-complete
(c) polynomially solvable (d) none of the above.

(ix) Chordal graph is a graph
(a) with at least four cycles
(b) in which all cycles of four or more vertices have an edge which is not part of the cycle
(c) which is triangulated
(d) which is also a perfect graph.

(x) Ford-Fulkerson algorithm for computing the maxium flow in a capacited directed graph is a/an
(a) greedy algorithm
(b) approximation algorithm
(c) dynamic programming algorithm
(d) none of the above.

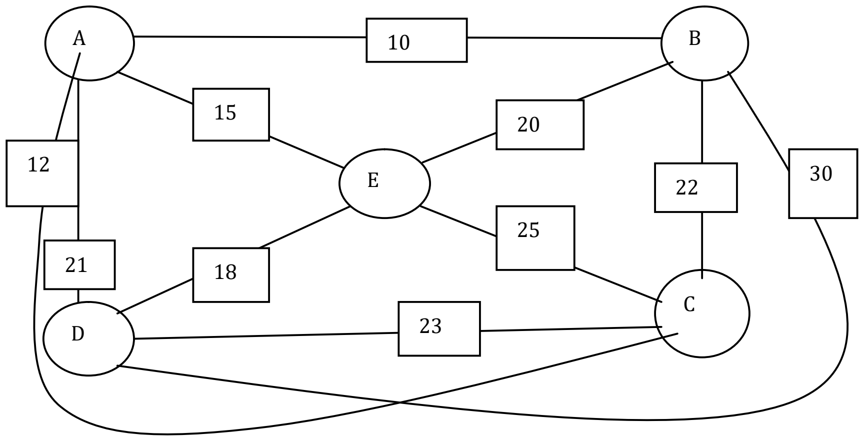
Group – B

2. (a) Analyze the time complexity of the Tarjan's algorithm for finding strongly connected components of a graph.
In the above algorithm, when the function to construct the Depth First Search tree of the given graph, is called for the second time, the vertices are considered in order of the decreasing finishing time. Why? Please explain all of the terms required.

(b) What is the condition for the existence of a closed Eulerian trail in a directed graph?
Prove that, in an even graph, every maximal trail must be closed.
Prove that, every non-loop edge has at least two vertices that are not cut vertices.

(3 + 3) + (1 + 3 + 2) = 12

3 (a) What is metric TSP? Prove that metric TSP is NP-complete.

(b) 

Find a TSP tour (T) in the above graph using dynamic programming such that T is less than or equal to double the optimal cost of the TSP tour. Justify your answer.

(1.5 + 3.5) + (5 + 2) = 12

Group – C

4. (a) Define matching, maximal matching and perfect matching.

   (b) Consider a typical run (i.e. the proposals are made by men) of Gale-Shapeley algorithm with n men and n women.

   (i) Prove that if a man m is free at some point in the execution of the algorithm, then there is a woman to whom he has not yet proposed.

   (ii) Show that the set S returned at termination is a perfect matching.

   (iii) Now that you have shown S to be a perfect matching, show that it is stable also.

(2 + 1 + 1) + (2 + 2 + 4) = 12

5. (a) State with an example, what an unstable pair is with respect to the stable matching algorithm.

   (b) How are the sizes of maximum independent set and minimum vertex cover of a graph connected?

   (c) State how the following three hard problems are connected – vertex cover problem, clique decision problem and maximum independent set problem.

   (d) (i) Write all the different perfect matchings in a cube (hypercube of order 3). Draw the cube first with the vertices labelled as a, b, c, d, e, f, g, and h.

   (ii) Can you give an example of a matching that is maximal but not perfect in the said graph?

3 + 2 + 3 + (3 + 1) = 12

Group – D

6. (a) What is a perfect graph? Give an example of a perfect graph.

   (b) State the Perfect Graph Theorem.

   (c) Prove that for any odd antihole, the chromatic number = clique number + 1.
   Note that we need the general result where |V| = 2k + 1.

   (d) What is a graphic sequence?
   Prove that for any graphic sequence, all the numbers in the sequence cannot be distinct.
   Show that the sum of the numbers in any graphic sequence cannot be odd.

2 + 1 + 5 + (1 + 2 + 1) = 12

7. (a) (i) From Euler's formula n + r – e = 2, it follows that in a planar graph 3r/2 ≤ e ≤ 3n – 6, where n, e and r are the number of vertices, edges and regions (faces) respectively. Using the above result can you show that every planar graph must have a vertex of degree ≤ 5?

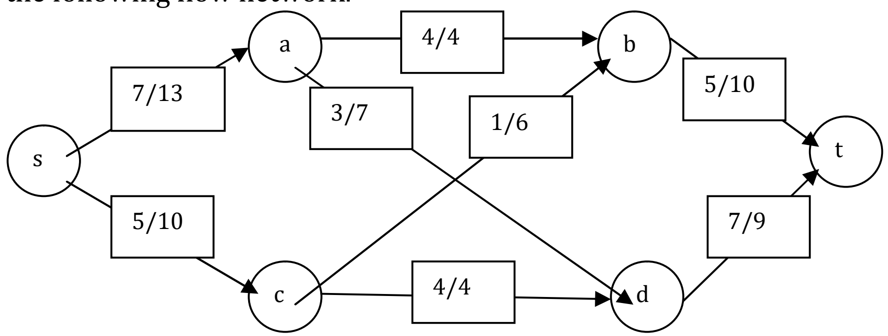
   (ii) Using the above result or otherwise show that at most five colors are needed to properly color a planar graph.

   (b) If all the vertices of a graph has degree ≤ 4, can we conclude that the graph must be planar? Answer with justification.

(2 + 7) + 3 = 12

Group – E

8. (a) Define augmenting path of a flow network. Draw the residual graph of the following flow network:



Find an augmenting path in the residual graph. Show the necessary changes in the above network and in the residual graph.

(b) Prove the following lemma:
A random graph $G_{n, p}$, given that its number of edges is m, is equally likely to be one of the ((n choose 2) choose m) graphs that have m edges.

(2 + 3 + 1) + 6 = 12

9. (a) Ford Fulkerson algorithm has following limitation:
Termination of the algorithm in a network with small number of nodes, even with integral flow can take large number of steps and depends on path selection and capacity of flow network.
Illustrate the above with a suitable example.

(b) Define interval graph and perfect elimination ordering (P.E.O) with a proper example.

(c) Prove that there is an efficient algorithm to solve graph colouring problem on graphs with P.E.O.

3 + (3 + 3) + 3 = 12