7. (a) Compute the QR decomposition of the following matrix using Gram-Schmidt process :

$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & 3 & 3 \\ -1 & -1 & 5 \\ 1 & 3 & 7 \end{pmatrix}$$

(b) Find the projections of the vector $(1, 2, 3, 1)$ along the basis vectors of the following basis $\{(1, 0, 0, 1), (-1, 0, 2, 1), (1, 3, 1, -1), (1, -1, 1, -1)\}$.

8 + 4 = 12

8. (a) Let $F: \mathbb{R}^4 \to \mathbb{R}^3$ be a linear mapping defined by $F(x,\ y,\ z,\ t) = (x - y + z + t, x + 2z - t, x + y + 3z - 3t)$. Find a basis and the dimension of (i) Image of F, (ii) Kernel of F.

(b) Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear mapping. Then prove that T is injective if and only if Ker $T = \{\theta\}$, θ being the null vector in V.

7 + 5 = 12

9. (a) State the Rank-Nullity theorem for linear transformation and verify it for the following linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T\ (x, y)\ =\ (2x + 3y, 4x - 5y)$.

(b) Show that the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3)\ =\ (bx_3 - cx_2,\ cx_1 - ax_3,\ ax_2 - bx_1)$ is not invertible where a, b, c are constants.

7 + 5 = 12

LINEAR ALGEBRA
(MATH 4182)

Time Allotted : 3 hrs **Full Marks : 70**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and <u>any 5 (five)</u> from Group B to E, taking <u>at least one</u> from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A
(Multiple Choice Type Questions)

1. Choose the correct alternative for the following: **10 × 1 = 10**

(i) An $n \times n$ matrix A is non-singular if and only if

(a) 0 is not an eigenvalue of A (b) $det\ A\ =\ 0$
(c) All the eigenvalues of A are distinct (d) 0 is an eigenvalue of A.

(ii) In a real quadratic form, the matrix associated with the real quadratic form is a real

(a) symmetric matrix (b) skew-symmetric matrix
(c) singular matrix (d) non-singular matrix.

(iii) The eigenvalues of a 2×2 idempotent matrix are

(a) 0, 1 (b) 1, -1 (c) 0, 0 (d) 1, 1 .

(iv) Let $f(t) = 3t - 5$ and $g(t) = t^2$, be two polynomials with inner product defined as $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$. Then $\langle f, g \rangle$ is

(a) $\frac{9}{10}$ (b) $-\frac{9}{10}$ (c) $-\frac{11}{12}$ (d) $\frac{1}{12}$.

(v) Consider the vectors $u = (2,3,5)$ and $v = (1, -4, 3)$ in $\mathbb{R}^3$. The angle between the two vectors is

(a) $\cos^{-1}(\frac{25}{\sqrt{988}})$ (b) $\cos^{-1}(\frac{5}{\sqrt{988}})$
(c) $\cos^{-1}(\frac{\sqrt{5}}{\sqrt{988}})$ (d) $\cos^{-1}(\frac{1}{\sqrt{988}})$.

(vi) If W be a subspace of an n −dimensional vector space V, then

(a) $\dim W > n$ (b) $\dim W \leq n$
(c) $\dim W = n$ (d) $\dim W \geq n$.

(vii) If A be an $n \times n$ matrix with diagonal entries as $\lambda_1, \lambda_2, \ldots, \lambda_n$ and rest of the entries as 0, then the roots of the equation $\det(A - xI) = 0$ are
(a) all equal to 1 (b) all equal to zero
(c) $\lambda_i, 1 \le i \le n$ (d) $-\lambda_i, 1 \le i \le n$.

(viii) Which of the following is not a linear transformation on $\mathbb{R}^3$:
(a) $T(x,y,z) = (x, 2y, 3x - y)$ (b) $T(x,y,z) = (0,0,0)$
(c) $T(x,y,z) = (y, x, 2z - y)$ (d) $T(x,y,z) = (1, x, z)$.

(ix) The eigenvalues of a 3×3 real matrix A are $(1, -2, 3)$. Then
(a) $A^{-1} = \frac{1}{6}(5I + 2A - A^2)$ (b) $A^{-1} = \frac{1}{6}(5I - 2A + A^2)$
(c) $A^{-1} = \frac{1}{6}(5I - 2A - A^2)$ (d) $A^{-1} = \frac{1}{6}(5I + 2A + A^2)$

(x) Which of the following statement is true?
(a) Any two bases of a finite dimensional vector space may or may not have same number of linearly independent vectors.
(b) Existence of a basis for a finite dimensional vector space is not necessary.
(c) If V be a vector space of dimension n (> 0), then any n linearly independent set of vectors of V is a basis of V.
(d) If V be a vector space of dimension n, then $(n - 1)$ linearly independent vectors will always generate V.

Group – B

2. (a) Find the Singular Value Decomposition of $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 2 & 2 \end{bmatrix}$.

(b) (i) For which values of constants a, b and c is the matrix $A = \begin{bmatrix} 7 & a & b \\ 0 & 2 & c \\ 0 & 0 & 3 \end{bmatrix}$ diagonalizable?

(ii) Is every diagonalizable matrix invertible? Justify your answer.

8 + (2 + 2) = 12

3. (a) Determine all eigenvalues and their algebraic multiplicities of the matrix $A = \begin{bmatrix} 1 & a & 1 \\ 1 & 1 & a \\ 1 & a & 1 \end{bmatrix}$, where a is a real number.

(b) Suppose A is a diagonalizable $n \times n$ matrix and has only 1 and -1 as eigenvalues. Show that $A^2 = I_n$, where I_n is the $n \times n$ identity matrix.

6 + 6 = 12

Group - C

4. (a) (i) Examine if the set of vectors $\{(1,2,4), (2,-1,3), (0,1,2)\}$ is linearly independent.

(ii) Find the dimension and a basis of the null space of the matrix A where,

$$A = \begin{pmatrix} 1 & 2 & -1 & 2 \\ 3 & 1 & 2 & 2 \\ 2 & -1 & 3 & 0 \\ 1 & -2 & 4 & -2 \end{pmatrix}$$

(b) Show that the set S is a subspace of the vector space $C[0,1]$ where $S = \{f \in C[0,1] : f(0) = 0\}$.

(3 + 6) + 3 = 12

5. (a) Find a basis and the dimension of the solution space W of the following system of equations:

$$x + 2y - 3w = 1$$
$$2x + 4y + 3z + w = 3$$
$$3x + 6y + 4z - 2w = 4$$

(b) If $\{\alpha, \beta, \gamma\}$ is a basis of a real vector space V and c is a non-zero real number, prove that $\{\alpha + c\beta, \beta, \gamma\}$ is also a basis of V.

7 + 5 = 12

6. (a) Define orthogonality of a vector space. Apply Gram-Schimdt orthogonalisation process to find an orthogonal basis for the subspace of $\mathbb{R}^4$ spanned by
$v_1 = (1,1,1,1)$, $v_2 = (1,2,4,5)$, $v_3 = (1,-3,-4,-2)$.

(b) Find the best approximate solutions of the following system of equations:

$$2x + y = 4$$
$$3x + 2y = 7$$
$$x - y = 3$$

7 + 5= 12