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# On Invex Functions in Hilbert Space

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**Abstract**

In this paper invex functions have been introduced in Hilbert space. Some important results regarding the characterization of such functions have been discussed. It has been proved that although being a generalization of the class of convex functions, this class of functions posses some properties which are not true in case of the class of convex functions in general.

***Keywords***: *Convexity, Invexity, Frechet Derivative, Archimedean Order*

***Mathematics Subject Classification:*** *26B25, 26A51 ; 49J50, 49J52*

## 1. Introduction

The mathematics of Convex Optimization was discussed by several authors for about a century [2, 3, 4, 5, 9, 10, 15, 17, 23, 24]. In the second half of the last century, various generalizations of convex functions have been introduced [2, 3, 4, 5, 6, 7, 10, 11, 12, 14, 16, 18, 19, 20, 22]. The *invex (invariant convex)*, *pseudoinvex* and *quasiinvex functions* were introduced

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by M.A.Hanson in 1981 [14]. These functions are extremely significant in optimization theory mainly due to the properties regarding their global optima. For example, *a differentiable function is invex iff every stationary point is a global minima*[6]. Later in 1986, Craven defined the non-smooth invex functions [11]. For the last few decades, generalized monotonicity, duality and optimality conditions in invex optimization theory have been discussed by several authors but mainly in $\mathbb{R}^n$[6,11,12,14,18,19,20]. The basic difficulty of genaralizing the theory in infinite dimensional spaces is that, unlike the case in finite dimension, closedness and boundedness of a set does not imply the compactness. However, in reflexive Banach spaces the problem can be alleviated by working with weak topologies and using the result that the closed unit ball is weakly sequentially compact.

In this paper, the concept of invex functions has been introduced in Hilbert space. Some important theorems regarding the characterization of such functions have been proved. It has been observed that the proposed class of invex functions posses some useful properties which do not hold for convex functions in general.



## 2. Prerequisites

**Definition 2.1:** [4] A subset C of $\mathbb{R}^n$ is convex if for every pair of points x1,x2 in C, the line segment

$$[x_1, x_2] = \{x : x = \alpha x_1 + \beta x_2, \alpha \geq 0, \beta \geq 0, \alpha + \beta = 1\} \tag{1}$$

belongs to C.

**Definition 2.2:** [21]The set C is said to be invex if there is a vector function $\eta : C \times C \to \mathbb{R}^n$ such that,

$$x_1 + \lambda\eta(x_1, x_2) \in C \quad \forall\, x_1, x_2 \in C \text{ and } \forall \lambda \in [0,1] \tag{2}$$

**Definition 2.3:** [4] Let C be an open convex set in $\mathbb{R}^n$ and let f be real valued and differentiable on C. Then f is convex if

$$f(x) - f(y) \geq \langle \nabla f(y), x - y \rangle, \quad \forall\, x, y \in C \tag{3}$$

**Definition 2.4:** [20] The function $f$ is said to be *invex* if there is a vector function $\eta : C\times C \to \mathbb{R}^n$ such that,

$$f(x)-f(y)\geq \langle \nabla f(y),\eta(x,y)\rangle\,,\quad \forall\, x,y\in C \tag{4}$$

**Definition 2.5**: [8] Let $X$ and $Y$ be two normed vector spaces. A continuous linear transformation $A : X \to Y$ is said to be the *Fréchét derivative*(*F-derivative*) of $f : X \to Y$ at $x$ if for every $\varepsilon > 0$, $\exists\ \delta > 0$ such that,

$$\| f(x+h)-f(x)-Ah\|_Y\leq \varepsilon \| h\|_X \qquad \forall h \quad \text{with} \quad \| h\|_X\leq \delta \tag{5}$$

When the derivative exists it is denoted by $Df\,(x)$.
It is to be noted that, $\| f(x+th)-f(x)-Ath\|_Y\leq \varepsilon \| th\|_X =|t|\,\varepsilon \| h\|_X, \forall t$ with $\| th\|_x\leq \delta$ . Which implies that,

$$\left\| \frac{f(x+th)-f(x)}{t}-Ah\right\|_Y\leq \varepsilon \| h\|_X$$
$$\Rightarrow \lim_{t\to 0}\frac{f(x+th)-f(x)}{t}=Ah=Df(x)h$$

**Proposition 2.1:** [17]Let $X$ be a vector space and $Y$ be a normed space. Let $S$ be a transformation mapping an open Set $D \subset X$ into an open set $E \subset Y$ and let $P$ be a transformation mapping $E$ into a normed space $Z$. Put $T = PS$ and suppose $S$ is F-differentiable at $x \in D$ and $P$ is F-differentiable at $y = S(x) \in E$. Then $T$ is F- differentiable at x and $DT(x) = DP(y)\, DS(x)$.

**Remark:** [4] It is to be noted that in $\mathbb{R}^n$ , $Df(x) = \nabla f(x)$.

## 3. Invex Sets and Invex Functions

Let $H$ be a real Hilbert space and $K \subset H$ be a pointed closed convex cone, i.e., $K \cap - K = \{\theta\}$, with nonempty interior. Let us consider the following notations for $x, y \in H$:

$$x \leqq_K y \Leftrightarrow y-x\in K$$
$$x \leqslant_K y \Leftrightarrow y-x\in K\setminus\{\theta\}$$
$$x <_K y \Leftrightarrow y-x\in int\, K$$

**Definition 3.1:** A set $S \subset H$ is said to be η− *invex* if there exist a vector function η: S x S → $H$ such that

$$x + \lambda\eta(x, y) \in S, \quad \forall x, y \in H$$

and for all λ ∈[0,1]

Let $H_1$ and $H_2$ be two real Hilbert spaces, $K_2$ be a pointed closed convex cone in $H_2$ and $I \subseteq H_1$ is an open invex set. Further, suppose that, $f : I \to H_2$ be F-differentiable.

**Definition 3.2:** The function *f* is said to be η–*invex* if there exist a vector function η: $I \times I \to H_1$ such that,

$$Df(y)\eta(x, y) \leq_{K_2} f(x) - f(y), \quad \forall\, x, y \in I \tag{6}$$

It is to be noted that $Df(y)$ η $(x,y)$ denotes the value obtained by operating F-differential of *f* at $y \in I$ on $\eta(x,y) \in H_1$. Henceforth, unless η needs to be specifically mentioned, we would simply say that, *f* is *invex*.

**Remark**: If $H_1 = \mathbb{R}^n$, $H_2 = \mathbb{R}$ and $\eta(x,y) = (x–y)$, Definition 3.2. coincides with the definition of a real valued convex function in $\mathbb{R}^n$.

**Example 3.1:** Let us consider the function *f*: $L^2[0,1] \to L^2[0,1]$ defined as,

$$f(x(t)) = (x(t) - sinx(t)),\ x > 0,\ n \in \mathbb{N}$$

Clearly, f($x$) is non-convex in nature. But it can be verified that f($x$) is invex considering

$$\eta(x(t), y(t)) = \begin{cases} \dfrac{4sin\dfrac{x(t)-y(t)}{2}}{cosx(t)-1} & \text{if } x(t) \neq 2n\pi \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

## 4. Characterization of Invex Functions

**Theorem 4.1:** *Let $f : I \to H_2$ be F-differentiable. Then f is invex iff every stationary point is a global minimizer.*

***Proof:*** Let $f$ be invex and $y$ is stationary point of $f$. Then $Df(y) = \mathbf{0}$. Therefore,

$$\theta \leq_{K_2} f(x) - f(y), \ \forall\, x \in I \tag{8}$$

This implies that $y$ is a global minimizer of $f$ over $I$.

Conversely, let us assume that, every stationary point is a global minimizer. If $y$ is a stationary point, then (6) is obvious. Let us assume that $y$ is not a stationary point. By the definition of F-derivative, we have,

$$Df(y)\eta(x,y) = f(y+\eta(x,y)) - f(y) - \varepsilon(\eta(x,y)) \tag{9}$$

where, $\frac{\|\varepsilon(\eta(x,y))\|}{\|\eta(x,y)\|} \to 0$ as $\|\eta(x,y)\| \to 0$ Let $\eta(x,y)) = x - y$, then

$$Df(y)\eta(x,y) = f(x) - f(y) - \varepsilon(\eta(x,y)) \leq_{K_2} f(x) - f(y) \tag{10}$$

which implies that $f$ is invex.

**Theorem 4.2:** *Let $f : I \to H_2$ and $g : I \to H_2$ be F-differentiable invex functions such that either $Df(y) = -\lambda Dg(y)$ for some $\lambda > 0$ and $-\lambda[g(x) - g(y)] \leq_{K2} f(x) - f(y)$ or $Df(y) \neq -\lambda Dg(y)$ for any $\lambda > 0$. Then $f$ and $g$ are invex with respect to same $\eta(\cdot,\cdot)$.*

***Proof:*** Let us prove the theorem by contradiction. Let $f$ and $g$ be invex with respect to the same $\eta(\cdot,\cdot)$. Let us assume that there exist $x, y \in I$ and $\lambda > 0$ such that,

$$\begin{aligned} &Df(y) = -\lambda Dg(y) \\ &(f(x) - f(y)) <_{K_2} -\lambda(g(x) - g(y)) \end{aligned} \tag{11}$$

Now, since $f$ and $g$ are invex with respect to the same $\eta(\cdot,\cdot)$ we have,

$$\begin{aligned} &Df(y)\eta(x,y) \leq_{K_2} f(x) - f(y) \\ &Dg(y)\eta(x,y) \leq_{K_2} g(x) - g(y) \end{aligned} \tag{12}$$

From (10) and (11) we have,

$$\begin{aligned} \theta &= (Df(y) + \lambda Dg(y))\eta(x,y) \\ &= (Df(y)\eta(x,y)) + \lambda(Dg(y)\eta(x,y)) \\ &\leq_{K_2} (f(x) - f(y)) + \lambda(g(x) - g(y)) \end{aligned} \tag{13}$$

Which contradicts the assumption (10).

It is to be mentioned here that if $H_1 = H_2 = \mathbb{R}^n$, then using Gale's Theorem of the alternatives for linear inequalities, we can very easily prove that the above conditions are necessary as well.

**Example 4.1:** The functions $f(x) = -2x^2$ and $g(x) = log(x)$ are invex with respect to same $\eta(\cdot,\cdot)$. One of the several choice for $\eta(\cdot,\cdot)$ is $x - y$.

## 5. Some Basic Operations

**Theorem 5.1:** *Let $f : I \subset H_1 \to H_2$ is invex with respect to $\eta(\cdot,\cdot)$ and $\psi : H_2 \to \mathbb{R}$ be a monotonic increasing differentiable convex function, then the composite function $\psi \circ f$ is invex with respect to the same $\eta(\cdot,\cdot)$.*

***Proof:*** Since $\psi$ is convex, we know that

$D\psi(h_1)(h_2 - h_1) \leq \psi(h_2) - \psi(h_1)), h_1, h_2 \in B_2$. Let $h_1 = f(x_1)$ and $h_2 = f(x_2)$ for some $x_1, x_2 \in I$. Then, we have,

$$D\psi f((x_1))(f(x_2) - f(x_1)) \leq \psi(f(x_2) - \psi(f(x_1)) \tag{14}$$

Now using Proposition 2.1 and (13) we obtain,

$$\begin{aligned} D(\psi \circ f)(x_1)(\eta(x_1, x_2) &= D\psi(f(x_1)) \circ Df(x_1)(\eta(x_1, x_2) \\ &\leq_{K_2} D\psi(f(x_1))(f(x_2) - f(x_1)) \\ &\leq_{K_2} \psi(f(x_2)) - \psi f((x_1)) \\ &= \psi \circ f(x_2) - \psi \circ f(x_1) \end{aligned} \tag{15}$$

which implies that $\psi \circ f$ is invex with respect to $\eta(\cdot,\cdot)$.

**Theorem 5.2:** *Let $f_1, f_2, \ldots, f_n : I \subset H_1 \to H_2$ are continuous and F-differentiable real valued invex functions with respect to same $\eta(\cdot,\cdot)$ at $x_0 \in I$.* Then $f(x) = \min_{1 \leq i \leq n} f_i(x)$ is also invex at $x_0$ with respect to same $\eta(\cdot,\cdot)$.

***Proof:*** Let $\Delta = \{i \in \{1,2,.........,n\} : f_i(x_0) = \min_{1\le j\le n} f_j(x_0)\}$ . Then we have for $i \in \Delta$ and for all $j \notin \Delta$ , $f_i(x_0) <_{K2} f_j(x_0)$. Since the functions $f_1, f_2, ....., f_n$ are continuous, there exist a neighbourhood $N_\delta(\theta)$ such that for every $x \in N_\delta(\theta)$, $i \in \Delta$, $j \notin \Delta$

$$f_i(x_0 + x) <_{K_2} f_i(x_0) + \frac{\varepsilon}{3} <_{K_2} f_j(x_0) - \frac{\varepsilon}{3} <_{K_2} f_j(x_0 + x)$$

Which implies that,

$$\min_{1\le i\le n} f_i(x_0 + x) = \min_{i\in\Delta} f_i(x_0 + x) \tag{16}$$

Now, since every F-differentiable function is G-differentiable, we have,

$$\begin{aligned} Df(x_0)h &= \lim_{t\to 0} \frac{f(x_0 + th) - f(x_0)}{t} \\ &= \lim_{t\to 0} \frac{\min_{1\le i\le n} f_i(x_0 + th) - \min_{1\le i\le n} f_i(x_0)}{t} \\ &= \lim_{t\to 0} \frac{\min_{i\in\Delta} f_i(x_0 + th) - \min_{i\in\Delta} f_i(x_0)}{t} \\ &= \lim_{t\to 0}\min_{i\in\Delta} \frac{f_i(x_0 + th) - f_i(x_0)}{t} \\ &= \min_{i\in\Delta} Df_i(x_0)h \end{aligned}$$

Therefore we can say,

$$Df(x_0) = \min_{i\in\Delta} Df_i(x_0) \tag{17}$$

Now by the invexity of $f_i$ , $\forall x \in x_0 + N_\delta(0)$ and $\forall i \in \Delta$ we have,

$$\begin{aligned} & Df_i(x_0)\eta(x, x_0) \le_{K_2} f_i(x) - f_i(x_0) \\ & \Rightarrow \min_{i\in\Delta} Df_i(x_0)\eta(x, x_0) \le_{K_2} f_i(x) - f_i(x_0) \\ & \Rightarrow \min_{i\in\Delta} Df_i(x_0)\eta(x, x_0) \le_{K_2} f_i(x) - \min_{i\in\Delta} f_i(x_0) \\ & \Rightarrow \min_{i\in\Delta} Df_i(x_0)\eta(x, x_0) \le_{K_2} \min_{i\in\Delta} f_i(x) - \min_{i\in\Delta} f_i(x_0) \end{aligned} \tag{18}$$

From (10) and (11) we get,

$$Df(x_0)\eta(x,x_0) \leq_{K_2} f(x)-f(x_0)$$

Note that, the above theorem does not hold for convex functions. The following example is an illustration to that:

**Example 5.1:** Consider $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ such that $f_1(x) = (x + a)^2$ and $f_2(x) = (x - a)^2$ , $a \in \mathbb{R}$ . it is obvious that both $f_1$ and $f_2$ are convex, but the following function is non-convex:

$$f(x) = \min\{f_1(x), f_2(x)\} = \begin{cases} (x+a)^2, & x \leq 0 \\ (x+a)^2, & x > 0 \end{cases}$$

However $f$ is invex at each $x \in \mathbb{R}$ with respect to $\eta(x, y) = x - y$.

## 6. Some Generalizations

**Definition 6.1:** The F-differentiable function $f : I \to H_2$ is *pseudoinvex* if there exist $\eta(\cdot,\cdot) : I \times I \to H_1$ and such that, for all $x, y \in I$ ,

$$\theta \leq_{K_2} Df(y)\eta(x,y) \Rightarrow \theta \leq_{K_2} f(x)-f(y) \tag{19}$$

$f$ is said to be *quasiinvex* if,

$$f(x)-f(y) \leq_{K_2} \theta \Rightarrow Df(y)\eta(x,y) \leq_{K_2} \theta \tag{20}$$

**Definition 6.2:** A differentiable function $f$ defined on an open set $S \subseteq H_1$ is called η-*pseudolinear* if $f$ and $-f$ are pseudo-invex with respect to the same η.

**Definition 6.3:** The function the function $\eta : S \times S \to H_1$ satisfies *Condition C*, if forevery $x, y \in S : \eta(y, y+\eta(x,y)) = -\lambda\eta(x,y)$ and $\eta(x, y+\eta(x,y)) = (1-\lambda)\eta(x,y)$ for all $\lambda \in [0,1]$.

**Theorem 6.1:** *Let $f$ be a differentiable function defined on an open invex set $I \subseteq H_1$ and $\eta : I \times I \to H_1$ satisfies Condition C. Suppose that $f$ is η-pseudolinear on I, then for all $x, y \in S$, $Df(y)\eta(x,y) = \theta$ iff $f(x) = f(y)$.*

***Proof:*** Suppose that $Df(y)\eta(x,y) = \theta$. Since $f$ is η-pseudolinear on I, $f$ and $-f$ both are pseudo-invex with respect to η. Therefore we have

$$\theta \leq_{K_2} Df(y)\eta(x,y) \Rightarrow \theta \leq_{K_2} f(x) - f(y)$$
$$Df(y)\eta(x,y) \leq_{K_2} \theta \Rightarrow f(x) - f(y) \leq_{K_2} \theta \tag{21}$$

Combining these inequalities, we obtain

$$Df(y)\eta(x,y) = \theta \Rightarrow f(x) - f(y) = \theta \tag{22}$$

Conversely, let us assume that $f(x) = f(x)$, $x, y \in I$. It can be very easily proved that this assumption implies $f(y + \lambda\eta(x,y)) = f(y)$, for all $\lambda \in (0.1)$.

If $f(y) <_{K_2} f(y + \lambda\eta(x,y))$, then by the considering the contra-positive statement of the definition of pseudo-invexity (12) for $-f$ with respect to η, we have

$$Df(y)\,\eta(y,z) \leq_{K_2} \theta \tag{23}$$

where $z = y + \lambda\eta(x,y)$.

Now, from Condition C we have, for $\lambda \in (0.1)$,

$$\eta(x,z) = (1-\lambda)\eta(x,y)$$
$$\eta(y,z) = -\lambda\eta(x,y) = -\frac{\lambda}{1-\lambda}\eta(x,z) \tag{24}$$

Therefore, from (22), we have $Df(z)(-\frac{\lambda}{1-\lambda}\eta(x,z)) \leq_{K_2} \theta$ and hence

$$\theta \leq_{K_2} Df(z)\,\eta(y,z) \tag{25}$$

Therefore, by pseudoinvexity of $f$, we have,

$$f(z) \leq_{K_2} f(x) \tag{26}$$

This contradicts the assumption that $f(x) = f(y) <_{K_2} f(y + \lambda\eta(x,y)) = f(z)$.

Similarly, using pseudo-invexity of $-f$, one can also show that the assumption $f(y+\lambda\eta(x,y)) < f(y), \forall\lambda \in (0,1)$, leads to a contradiction. This proves the claim that, $f(y+\lambda\eta(x,y)) = f(y), \forall \in (0,1)$. Now, we know that if F-differential exists then G-differential also exist and they are equal. Therefore,

$$Df(y)\eta(x,y) = \lim_{\lambda\to 0}\frac{f(y+\lambda\eta(x,y)-f(y)}{\lambda} = \theta$$

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