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Application of the Concept of Quasiinvexity on a Class of Semidefinite Minimization Problems

Sandip Chatterjee
Heritage Institute of Technology
Kolkata, 700107, West Bengal, India
functionals@gmail.com

R. N. Mukherjee
The University of Burdwan
Department of Mathematics
Bardhaman-713104, West Bengal, India
rnm_bu_math@yahoo.co.in

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Abstract

In this paper the concept of quasiinvexity has been introduced on a class of semidefinite minimization problems in the field of complex numbers. A global optimality condition for such problems has also been obtained.

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1 Introduction

In 1977, Zang, Choo and Avriel [3] studied functions whose stationary points are global minima. By considering the level sets of a real valued function as a point-to-set mapping and by examining its semicontinuity property, Zang, Choo and Avriel obtained the result that a real valued function, defined on a subset of $\mathbb{R}^n$ and satisfying some mild regularity conditions, belongs to the class of functions whose stationary points are global minima if and only if the point-to-set mapping of its level sets is strictly lower semicontinuous. A few years later, in 1981, M. A. Hanson[6] introduced the concept of invexity for differentiable functions in $\mathbb{R}^n$. Hanson actually generalized one of the most important properties of convex functions that they are always bounded on one side by their tangent hyperplanes at any point which facilitates the use of linear bounds and approximations. Hanson weaken the class of convex functions by introducing nonlinear bounds using the generalization of Taylor's expansion. Hanson's introduction of invexity has been a significant generalization of convex functions. Hanson's initial result inspired a great deal of subsequent work which has hugely expanded the role and applications of invexity in nonlinear optimization and other branches of pure and applied sciences. In 1985, Craven and Glover [2] showed that the class of invex functions is equivalent to the class of functions whose stationary points are global minima. This property played the pivotal role in the introduction of invex functions in optimization theory.

Several researchers have generalized the concept of invexity and found some interesting characteristics of such generalizations which has been very useful in the context of optimization and equilibrium problems. The concept of quasiinvex functions have been introduced by Pini [8]. Pini proved that a differentiable preinvex function is invex but not conversely. Later, Mohan and Neogy [14] redefined quasiinvex functions with some rectifications. Mohan and Neogy stiudied invex sets in detail and showed how to build such sets into $\mathbb{R}^n$ using invex sets in a lower dimensional space. In [11], Kaul and Kaur referred the invex, pseudoinvex and quasiinvex functions as η-convex, η-pseudoconvex and η-quasiconvex respectively. In [12, 13], Chatterjee and Mukherjee studied invex functions in Hilbert spaces and obtained the necessary and sufficient condition for the existence of global optimal solution of invex programming problems posed in an arbitrary Hilbert space.

Semidefinite Programming can be regarded as an extension of linear optimization problems which has many applications in different branches of science and engineering . Most of the interior point methods of linear programming has been generalized to semidefinite convex programming or semidefinite quasiconvex programming in the field of real numbers [4, 5, 7, 9, 10].

In this paper, a new class of optimization problems to be known as Semidefinite Quasiinvex Programming Problem has been introduced and necessary and sufficient condition for the existence of global optima of such class of problems has been discussed. The concept has been generalized in the field of complex numbers.

2 Prerequisites

Definition 2.1. [5] The *Frobenius norm* of a matrix $A = (a_{ij})_{m\times n}$ is defined as

$$\|A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{i=1}^{n} a_{ij}} = \sqrt{Tr(AA^H)} \tag{2.1}$$

where A^H is the transjugate of A.

Definition 2.2. [5] The *Frobenius product* of two matrices $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ is defined as

$$\langle A, B\rangle_F = Tr(A^H B) = Tr(B^H A) \tag{2.2}$$

It is obvious that $\|A\|_F = \sqrt{\langle A, A\rangle_F}$.
Definition 2.3. [9] A Semidefinite Minimization Problem is defined as follows:

$$\begin{aligned} &\text{Min}\langle C, X\rangle_F \\ &\text{subject to } \langle A_j, X\rangle_F \leq b_j \\ &j = 1, 2, 3, ..., m,\ X \succeq 0 \end{aligned} \tag{2.3}$$

where $C, X, A_j \in \mathbb{R}^{n\times n}$
Definition 2.4. [5] The function $f : \mathbb{C}^{n\times n} \to \mathbb{R}$ is said to be *coercive* if

$$\lim_{\|x\|\to\infty} f(x) = \infty \tag{2.4}$$

Definition 2.5. [10] The *level set* $E_{f(Z)}(f)$ of the function $f : \mathbb{C}^{n\times n} \to \mathbb{R}$ at a point $Z \in \mathbb{C}^{n\times n}$ is defined as

$$E_{f(Z)}(f) = \{Y \in \mathbb{C}^{n\times n} : f(Y) = f(Z)\} \tag{2.5}$$

Note. Let A and B be two Hermitian matrices, then $Tr(AB) \in \mathbb{R}$.

Let H be a real Hilbert space and $K \subset H$ be a pointed closed convex cone, i.e., $K \cap -K = \{\theta\}$, with nonempty interior. Let us consider the following notations for $x, y \in H$:

$$\begin{aligned} x \leqq_K y &\Longleftrightarrow y - x \in K \\ x \leqslant_K y &\Longleftrightarrow y - x \in K \setminus \{\theta\} \\ x <_K y &\Longleftrightarrow y - x \in intK \end{aligned}$$

Definition 2.6. [13] A set $S \subset H$ is said to be $\eta - invex$ if there exist a vector function $\eta : S \times S \to H$ such that

$$x + \lambda\eta(x, y) \in S, \quad \forall x, y \in H \text{ and for all } \lambda \in [0, 1]$$

Let H_1 and H_2 be two real Hilbert spaces, K_2 be a pointed closed convex cone in H_2 and $I \subseteq H_1$ is an open invex set. Further, suppose that, $f : I \to H_2$ be a F-differentiable.
Definition 2.7. [13] The function f is said to be $\eta - invex$ if there exist a vector function $\eta : I \times I \to H_1$ such that,

$$Df(y)\eta(x, y) \leq_{K_2} f(x) - f(y), \quad \forall\ x, y \in I$$

It is to be noted that $Df(y)\eta(x, y)$ denotes the value obtained by operating F-differential of f at $y \in I$ on $\eta(x, y) \in H_1$. If there is no confusion regarding η we would simply say that, f is *invex*.
Remark: If $H_1 = \mathbb{R}^n$, $H_2 = \mathbb{R}$ and $\eta(x, y) = (x - y)$, Definition 2.2.2. coincides with the definition of a real valued convex function in $\mathbb{R}^n$.
Example 2.2.1. [13] Let us consider the function $f : L^2[0, 1] \to L^2[0, 1]$ defined as,

$$f(x(t)) = (x(t) - sinx(t))\ ,\ x > 0,\ n \in \mathbb{N}$$

Clearly, f(x)is non-convex in nature. But it can be verified that f(x) is invex considering

$$\eta(x(t), y(t)) = \begin{cases} \dfrac{4sin\dfrac{x(t)-y(t)}{2}}{cosx(t)-1} & \text{if } x(t) \neq 2n\pi \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.5.1.[13] The F-differentiable function $f : I \to H_2$ is *pseudoinvex* if there exist $\eta(\cdot,\cdot) : I \times I \to H_1$ and such that, for all x, $y \in I$,

$$\theta \leq_{K_2} Df(y)\eta(x, y) \Rightarrow \theta \leq_{K_2} f(x) - f(y)$$

f is said to be *quasiinvex* if,

$$f(x) - f(y) \leq_{K_2} \theta \Rightarrow Df(y)\eta(x, y) \leq_{K_2} \theta$$

3 Real Valued Quasiinvex Matrix Functions on $\mathbb{C}^{n\times n}$

Definition 3.1. Let X be a matrix in $\mathbb{C}^{n\times n}$ and $f : \mathbb{C}^{n\times n} \to \mathbb{R}$. The derivative of f is defined by

$$f'(X) = (\tfrac{\partial f(X)}{\partial x_{ij}})_{n\times n}$$

If f is differentiable, $f'(X)$ is symmetric and H is Hermitian, then by Taylor's formula

$$f(X + H) - f(X) = \langle f'(X), H\rangle_F + O(\|H\|_F)$$

Definition 3.2. A set $D \subset \mathbb{C}^{n\times n}$ is $\eta - invex$ if there exist $\eta : \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ such that

$$Y + \alpha\eta(X, Y) \in D \text{ for all } X, Y \in D \text{ and } \alpha \in [0, 1]$$

Definition 3.3. The function $f : D \to \mathbb{R}$ is said to be $\eta - quasiinvex$ if

$$f(Y + \eta(X, Y)) \leq \max\{f(X), f(Y)\} \text{ for all } X, Y \in D$$

It is to be noted that f becomes a quasiconvex function when $\eta(X, Y)$ is chosen to be $X - Y$.

Definition 3.4. Let $\mathbb{H}^{n\times n}$ be the set of all $n \times n$ Hermitian matrices. The function $f : \mathbb{C}^{n\times n} \to \mathbb{R}$ is said to be $\eta - invex$ if $f'(\cdot)$ is symmetric and there exist $\eta(\cdot,\cdot) : \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n} \to \mathbb{H}^{n\times n}$ such that,

$$f(Y + \eta(X, Y)) \leq \langle f'(Y), \eta(X, Y)\rangle_F \text{ for all } X, Y \in H$$

It is to be noted that if $\eta(X, Y) = X - Y$, then f becomes convex. Henceforth, Unless η needs to be mentioned specifically, we would simply simply refer a function as quasiinvex or invex.

Theorem 3.1. A function $f : \mathbb{C}^{n\times n} \to \mathbb{R}$ is quasiinvex iff the lower level set $L_c(f) = \{X \in \mathbb{C}^{n\times n} : f(X) \leq c\}$ is invex for all $c \in \mathbb{R}$.

Proof: Let $c \in \mathbb{R}$ and $X, Y \in L_c(f)$. By the definition of quasiinvexity,we have,

$$f(Y+\alpha\eta(X,Y)) \le \max\{f(X), f(Y)\} \le c,\ \forall \alpha \in [0,1]$$

which implies that $Y+\alpha\eta(X,Y) \in L_c(f)$ i.e. $L_c(f)$ is invex. Conversely, let $L_c(f)$ is invex for all $c \in \mathbb{R}$. Define $c^* = \max\{f(X), f(Y)\}$. Then $X \in L_{c^*}(f)$ and $Y \in L_{c^*}(f)$. By the invexity of $L_{c^*}(f)$,

$$f(Y+\alpha\eta(X,Y)) \le c^* = \max\{f(X), f(Y)\}$$

Hence f is quasiinvex. □

Theorem 3.2. Let $f : \mathbb{C}^{n\times n} \to \mathbb{R}$ be a quasiinvex differentiable function. Then $f(X) \le f(Y)$ for $X, Y \in \mathbb{C}^{n\times n}$ implies that $\langle f'(Y), \eta(X,Y))\rangle_F \le 0$, provided $f'(Y)$ is symmetric and $\eta(X,Y)$ is Hermitian.

Proof: By the quasiinvexity of f,

$f(Y+\alpha\eta(X,Y)) \le \max\{f(X), f(Y)\} = f(Y),\ \forall \alpha \in [0,1]$

$\Rightarrow f(Y+\alpha\eta(X,Y)) - f(Y) \le 0$

$\Rightarrow \alpha\Big(\langle f'(Y), \eta(X,Y))\rangle_F + \frac{o(\alpha\|\eta(X,Y)\|_F}{\alpha}\Big) \le 0$

$\Rightarrow \langle f'(Y), \eta(X,Y))\rangle_F \le 0$ □

4 Semidefinite Quasiinvex Programming Problem on $\mathbb{C}^{n\times n}$

Let us denote the set of all $n \times n$ Hermitian matrices by $\mathbb{H}^{n\times n}$. Consider the following minimization problem

$$\begin{aligned} \text{Minimize } & f(X) \\ \text{subject } to\ & g_j(X) \le b_j \\ & X \succeq 0 \end{aligned} \tag{4.1}$$

where f,$g_j : \mathbb{H}^{n\times n} \to \mathbb{R}$, $j = 1, 2, ...,m$ are η-quasiinvex differentiable functions, such that $\eta(\cdot,\cdot)$ is Hermitian, $f'(\cdot)$ is symmetric and X is positive semidefinite. We call the problem as the *semidefinite quasiinvex programming problem.*

Example 4.1. Following is an example of a semidefinite quasiinvex programming problem.

$$\text{Minimize } f(X) = \|X\|_F^2$$
$$\text{Subject to } \|A\|_F \le \|X\|_F \le \|B\|_F$$
$$\text{where } A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix},\ B = \begin{pmatrix} 2 & 2i \\ -2i & 3 \end{pmatrix} \text{ and } X \succeq 0$$

Let $D = \{X \in \mathbb{R}^{n\times n} : g_j(X) \le b_j, j = 1, 2, 3, ...,m\}$. The following theorem provides a global optimality condition for the problem (1).

Theorem 4.1. Let Z be a solution of (1). Then $\langle f'(X), \eta(X,Y)\rangle_F \ge 0$, $\forall\, Y \in E_{f(Z)}(f)$, $X \in D$. Moreover, if f is coercive , quasiconvex and $f'(X+\alpha f'(X)) \ne 0$, $\forall X \in D$ and $\alpha \ge 0$, then the condition is sufficient also.

Proof: Let us assume that Z is a solution of (1). Let $X \in D$ and $Y \in E_{f(Z)}(f)$. Then we have $f(Y) - f(X) = f(Z) - f(X) \leq 0$. Therefore using Theorem 3.1 we can conclude $\langle f'(X), \eta(X,Y)\rangle$. Conversely, suppose that Z is not a solution of (1). Then there exist a $U \in D$ such that $f(U) < f(Z)$. For $\alpha > 0$, let us define $Y_\alpha = U + \alpha f'(U)$.

$$f\big(U + \alpha f'(U)\big) - f(U) = \alpha\Big(\|f'(U)\|^2 + \tfrac{o(\alpha\|f'(U)\|)}{\alpha}\Big)$$

Using Theorem 3.2. and the assumption, it can be proved that $f(Y_\alpha) > f(U)$ for all $\alpha > 0$. Now, since f is coercive, $f(Y_\alpha) \to \infty$ as $\alpha \to \infty$. Therefore there exist $\hat{\alpha}$ such that $f(Y_{\hat{\alpha}}) > f(Z) > f(U)$. Which implies that there exist $\bar{\alpha}$ such that $f(Y_{\bar{\alpha}}) = f(Z)$, i.e. $Y_{\bar{\alpha}} \in E_{f(Z)}(f)$. Since $\eta(\cdot,\cdot)$ is quasiconvex,

$$\langle f'(U), \eta(U, Y_{\bar{\alpha}})\rangle_F = \tfrac{1}{\bar{\alpha}}\langle Y_{\bar{\alpha}} - U, U - Y_{\bar{\alpha}}\rangle_F = -\tfrac{1}{\alpha}\|Y_{\bar{\alpha}} - U\|^2 < 0,$$

a contradiction. Hence Z must be a solution of (1). □

Note: For Example 4.1., it can be verified that $Z = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$.

In the next section, we are going to present a special case of (4.1), motivated by Enkhbat and Bayartugs [1].

5 Semidefinite Convex Programming Problems in $\mathbb{C}^{n\times n}$

Let us consider the following semidefinite convex program:

$$\min_{X \in D} \; f(X) \tag{5.1}$$

where $f : \mathbb{C}^{n\times n} \to R$ is strongly convex and continuously differentiable, and D is an arbitrary compact set in $\mathbb{H}^{n\times n}$. In this case, we can weaken Theorem 4.1. as stated in the following theorem.

Theorem 5.1. Let Z be a solution of (5.1). Then

$$\langle f'(X), X - Y\rangle \geq 0, \; \forall Y \in E_{f(Z)}(f), \; X \in D \tag{5.2}$$

If, in addition,

$$\min_{X \in D} \|f'(X)\|_F > 0 \tag{5.3}$$

holds, then the condition above is also sufficient.

Proof. Assume that Z is a solution of (5.1). Consider $X \in D$ and $Y \in E_{f(z)}(f)$. Then, by convexity of f, we have $0 \geq f(Z) - f(X) = f(Y) - f(X) \geq \langle f'(X), Y - X\rangle_F$.

Let us prove the sufficiency using the method of contradiction. Assume that (5.2) holds and there exist a point $U \in D$ such that $f(U) < f(Z)$. Clearly,$f'(U) \neq 0$ by assumption (5.3). Now define U_α as $U_\alpha = U + \alpha f'(U)$. Then, by convexity of f, we have

$$f(U_\alpha) - f(U) \geq \langle f'(U), U_\alpha - U\rangle_F = \alpha\|f'(U)\|_F^2 \tag{5.4}$$

which implies

$$f(U_\alpha) \geq f(U) + \alpha\|f'(U)\|_F^2 > f(U) \tag{5.5}$$

Then find $\alpha = \bar{\alpha}$ such that,

$$f(U) + \bar{\alpha}\|f'(U)\|_F^2 = f(Z) \tag{5.6}$$

i.e., $\bar{\alpha} = \frac{f(Z)-f(U)}{\|f'(U)_F^2} > 0$.
Thus, we get,

$$f(U_{\bar{\alpha}}) \geq f(U) + \bar{\alpha}\|f'(U)\|_F^2 = f(Z) > f(U) \tag{5.7}$$

Define a function $h : \mathbb{R}^+ \to \mathbb{R}$ as

$$h(\alpha) = f(U + \alpha f'(U)) - f(Z) \tag{5.8}$$

It is clear that h is continuous on $[0, +\infty]$. Note that $h(\bar{\alpha} \geq 0$ and $h(0) < 0$. There are two cases with respect to the values of $h(\bar{\alpha})$ which we should consider.
Case I: Let $h(\alpha) = 0$ (i.e., $f(U + \bar{\alpha})f'(U)) = f(Z)$, then

$$\begin{aligned}\langle f'(U), U - U_{\bar{\alpha}}\rangle_F &= -\langle f'(U), \bar{\alpha} f'(U)\rangle_F \\ &= -\bar{\alpha}\|f'(U)\|_F^2 < 0\end{aligned}$$

contradicting condition (5.3).
Case-II: Let $h(\bar{\alpha}) = 0$ and $h(0) < 0$. Since, h is continuous, there exists a point $\alpha_0 \in (0, \bar{\alpha}$ such that $h(\alpha_0) = 0$, i.e., $f(U + \alpha_0 f'(U)) = f(Z)$. Then we have

$$\langle f'(U), U - U_0\rangle_F = -\alpha_0\|f'(U)\|_F^2 < 0 \tag{5.9}$$

again contradicting (5.3).
Thus, in both the cases we find contradictions which proves the theorem. □
Let us define,
$\psi(Z) = \min\limits_{Y \in E_{f(Z)}(f)} P(Y),\ Z \in D$, where $P(Y) = \min\limits_{X \in D}\langle f'(X), X - Y\rangle_F,\ Y \in \mathbb{H}^{n\times n}$.
On the basis of Theorem 5.1. and the definition of $\psi(Z)$ let us generalize the *Algorithm MIN*, proposed by Enkhbat and Bayartugs [10], for a strongly convex function f such that $f'(\cdot)$ is symmetric and a compact set D :
Algorithm MIN:
Step I: Choose a feasible point $X^\circ \in D$. Set $k = 0$.
Step II: Solve the following problem:

$$\min_{Y \in E_{f(X_K)}(f)} P(Y)$$

Let Y_k be a solution of this problem (i.e.,$P(Y_k) = \min\limits_{X \in D}\langle f'(X), X - Y_k\rangle_F = \min\limits_{Y \in E_{f(X_k)}(f)} P(Y)$)
and let $\psi(X_k) = P(Y_k) = \langle f'(X_{k+1}), X_{k+1} - Y_k\rangle_F$.
Step III: If $\psi(X_k) = 0$ then the solution is X_k, otherwise, let k=k+1 and return to Step II.
The convergence of the algorithm is based on the following theorem:
Theorem 5.2. Assume that $f : \mathbb{H}^{n\times n} \to \mathbb{R}$ is strongly convex and continuously differentiable and D is a compact set in $\mathbb{C}^{n\times n}$. Let $\min_{X\in D}\|f'(X)\| > 0$. Then the sequence $\{X_k, k = 0, 1, ...\}$ generated by Algorithm MIN is a minimizing sequence for problem (5.2)

and every accumulation point of the sequence $\{X_k\}$ is a global minimizer of (5.1).
Proof. From the construction of X_k, we have $X_k \in D$ and $f(X_k) \geq f_*$ for all k, where $f_* = f(X_*) = \min_{X \in D} f(X)$. Clearly, $f'(X_*) \neq 0$ by assumption. Also, note that for all $Y \in E_{f(X_k)(f)}$ and $X \in D$, we have
$\psi(X_k) = \min_{Y \in E_{f(X_k)}(f)} \min_{X \in D} \langle f'(X), X - Y \rangle_F \leq \langle f'(X), X - Y \rangle_F \leq 0$ Now, if there exist a k such that $\psi(X_k) = 0$ then the proof is complete.
Therefore, let us assume that $\psi(X_k) < 0$ for all k and prove the theorem by contradiction. Let us suppose that X_k is not a minimizing sequence for the problem (5.1). Then,

$$\lim_{k \to \infty} \inf f(X_k) > f_* \tag{5.10}$$

By the definition of $\psi(X_k)$ and *Algorithm MIN*, we have

$$P(Y_k) = \psi(X_k) = \min_{Y \in E_{f(X_k)}(f)} \min_{X \in D} \langle f'(X), X - Y \rangle_F = \langle f'(X_{k+1}), X_{k+1} - Y_k \rangle_F \tag{5.11}$$

and by the definition of level set we have $f(Y_k) = f(X_k)$. The convexity of f implies that

$$f(X_k) - f(X_{k+1}) = f(Y_k) - f(X_{k+1}) \geq \langle f^(X_{k+1}), Y_k - X_{k+1} \rangle_F = -\psi(X_k) > 0 \tag{5.12}$$

Hence we obtain $f(X_{k+1}) < f(X_k)$ for all k, and the sequence $\{f(X_k)\}$ is strictly decreasing. Since the sequence is bounded below by f_*, it is convergent and satisfies

$$\lim_{k \to \infty} (f(X_{k+1} - f(X_k)) = 0 \tag{5.13}$$

Now from (16) and (17) we have

$$\lim_{k \to \infty} \psi(X_k) = 0 \tag{5.14}$$

From (18) we have $f(X_k) > f(X_*)$ for all k. Now define a ray V_α as follows:

$$V_\alpha = X_* + \alpha f'(X_*), \ \alpha > 0 \tag{5.15}$$

Then by the convexity of f, we have

$$f(V_\alpha) - f(X_*) \geq \langle f'(X_*), V_\alpha - X_* \rangle_F = \alpha \|f'(X_*)\|^2 \tag{5.16}$$

which implies that,

$$f(V_\alpha) \geq f(X_*) + \alpha \|f'(X_*)\|^2 > f(x_*), \ \alpha > 0 \tag{5.17}$$

Choose $\alpha = \alpha_k$ such that

$$f(X_*) + \alpha_k \|f'(X_*)\|^2 > f(X_k) \tag{5.18}$$

i.e.,

$$\alpha_k > \frac{f(X_k) - f(X_*)}{\|f'\|^2} > 0 \tag{5.19}$$

Now, define a function $h_k : \mathbb{R}^+ \to \mathbb{R}$ as

$$h_k(\alpha) = f(X_* + \alpha f'(X_*)) - f(X_k) \tag{5.20}$$

It is quite obvious that h_k is continuous on $[0,\infty)$. Note that $h_k(\alpha_k) > 0$ and $h_k(0) < 0$. Since h_k is continuous, there exist a point $\bar{\alpha_k} \in (0, \alpha_k)$ such that $h_k(\bar{\alpha_k}) = 0$,i.e., $f(V_{\bar{\alpha_k}}) = f(X_k)$ and $V_{\bar{\alpha_k}} = X_* + \bar{\alpha_k} f'(X_*)$.It is further to be noted that

$$\psi(X_k) = \min_{Y \in E_{f(X_k)}(f)} \min_{X \in D} \langle f'(X), X - Y \rangle_F \leq \langle f'(X_*), X_* - V_{\bar{\alpha_k}} \rangle_F \tag{5.21}$$

Taking into account $V_{\bar{\alpha_k}} = X_* + \bar{\alpha_k} f'(X_*)$, we have,

$$\begin{aligned} -\psi(X_k) \geq \langle f'(X_*), V_{\bar{\alpha_k}} - X_* \rangle_F &= \|f'(X_*)\| \|V_{\bar{\alpha_k}} - X_*\| \\ &\geq \min_{X \in D} \|f'(X)\| \|V_{\bar{\alpha_k}} - X_*\| > 0 \end{aligned} \tag{5.22}$$

Since $\psi(X_k) \to 0$ as $k \to \infty$ we have,

$$\lim_{k \to \infty} V_{\bar{\alpha_k}} = X_* \tag{5.23}$$

Now, the continuity of f implies,

$$\lim_{k \to \infty} f(X_k) = \lim_{k \to \infty} f(V_{\bar{\alpha_k}}) = f(X_*) \tag{5.24}$$

which is a contradiction to (6.11). Consequently, $\{X_k\}$ is a minimizing sequence for the problem (6.2). Now, since D is compact, there exist a subsequence $\{X_{k_n}\}$ which converges to $\bar{X}$ (say) in D such that,

$$\lim_{n \to \infty} f(X_{k_n}) = f(\bar{X}) = f_* \tag{5.25}$$

which completes the proof. □

6 Conclusion

Several authors have discussed numerous properties of semidefinite convex programming problems. There exists some nice computer oriented algorithms also to solve such kind of problems. In this paper, the concept of convexity has been generalized to the concept of invexity for semidefinite programming problems. The underlying field has also been generalized as a field of complex numbers. As a result, the class of optimization problems for which the stationary point is a global minima, has been weakened further. Theorem 5.2. ensures the existence of a solution for such class of problems. Hence a computer oriented algorithm for the semidefinite quasiinvex programming problems can also be designed, possibly when $\eta(\cdot, \cdot)$ is fixed.

References

[1] B.D. Craven, Nondifferentiable optimization by smooth approximations, *Optimization* **17** (1986), 3 - 17.

[2] B.D. Craven, B.M. Glover, Invex functions and duality, *Journal of Australian Mathematical Society* **24** (1985), 1 - 20.

[3] I. Zang, E.U. Choo, M. Avriel, On functions whose stationary points are global minima, *Journal of Optimization Theory and Applications* **22** (1977), 195-208.

[4] L. Monique, R. Franz, *Semidefinite Programming and Integer Programming, Report NAR0210* (2002), CWI, Amsterdam.

[5] L. Vandenberghe, S. Boyd, Semidefinite Programming, *SIAM Review* **38** (1996), 49 - 95.

[6] M.A. Hanson, On Sufficiency of Kuhn-Tucker Conditions, *Journal of Mathematical Analysis and Applications* **80** (1981), 545-550.

[7] P.M. Pardalos, H. Wolkowicz H (Eds.), Topics in Semidefinite and Interior Point Methods, *Fields Institute Communications 18, American Mathematical Society* (1998), Rhode Island.

[8] R. Pini R, Invexity and Generalized Convexity, *Optimization* **22**(4) (1991), 513 - 525.

[9] R. Enkhbat, Quasiconvex Programming, *Lambert Publisher* (2009), Berlin.

[10] R. Enkhbat, T. Bayartugs, Quasiconvex Semidefinite Minimization Problem, *Journal of Optimization* (2013), Article ID 346131, http://dx.doi.org/10.1155/2013/346131.

[11] R.N. Kaul, S. Kaur, Generalizations of convex and related functions, *European Journal of Operational Research* **9** (1982), 369377.

[12] Sandip Chatterjee, R.N. Mukherjee, On Invex Programming Problem in Hilbert Spaces, *Yugoslav Journal of Operations Research* (2015), DOI: (10.2298/YJOR141015010C).

[13] Sandip Chatterjee, R.N. Mukherjee, Invexity and a Class of Constrained Optimization Problems in Hilbert Spaces. *Facta Universitatis (Niš), Series: Mathematics and Informatics* **29**(4) (2015), 337 - 342.

[14] S.R. Mohan, S.K. Neogy, On invex sets and preinvex functions, *Journal of Mathematical Analysis and Applications* **189** (1994), 901 - 908.