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ON INVEX PROGRAMMING PROBLEM IN HILBERT SPACES

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Abstract: In this paper we introduce the invex programming problem in Hilbert space. The requisite theory has been established to characterize the solution of such class of problems.

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1. INTRODUCTION

The mathematics of Convex Optimization was discussed by several authors for about a century [2, 3, 4, 5, 9, 10, 15, 17, 23, 24]. In the second half of the last century, various generalizations of convex functions have been introduced [2, 3, 4, 5, 6, 7, 10, 11, 12, 14, 16, 18, 19, 20, 22]. The *invex(invariant convex), pseudoinvex* and *quasiinvex functions* were introduced by M.A.Hanson in 1981 [14]. These functions are extremely significant in optimization theory mainly due to the properties regarding their global optima. For example, *a differentiable function is invex iff every stationary point is a global minima*[6]. Later in 1986, Craven defined the non-smooth invex functions [11]. For the last few decades generalized monotonicity, duality and optimality conditions in invex optimization theory have been discussed by several authors but mainly in $\mathbb{R}^n$ [6, 11, 12, 14, 18, 19, 20]. The basic difficulty of genaralizing the theory in infinite dimensional spaces is that, unlike the case in finite dimension, closedness and boundedness of a set does not imply the compactness. However, in reflexive Banach spaces the problem can be alleviated by working with weak topologies and using the result that the closed unit ball is weakly sequentially compact.

In this paper the concept of invex functions and a class of optimization problems involving invex functions have been introduced in Hilbert spaces. A generalization of the very well known Fritz-John conditions regarding the existence of optimal solutions of such problems has been proposed. It has been proved that under the assumption of invexity, these conditions are not only necessary but also sufficient.

2. PREREQUISITES

Definition 2.1. A subset C of $\mathbb{R}^n$ is *convex* [4] if for every pair of points x_1, x_2 in C, the line segment

$$[x_1, x_2] = \{x : x = \alpha x_1 + \beta x_2,\ \alpha \geq 0,\ \beta \geq 0, \alpha + \beta = 1\}$$

belongs to C.

The set C is said to be *invex* [21] if there is a vector function $\eta : C \times C \to \mathbb{R}^n$ such that

$$x_1 + \lambda\eta(x_1, x_2) \in C \quad \forall\ x_1, x_2 \in C \ and\ \forall \lambda \in [0, 1]$$

Definition 2.2. Let C be an open convex set in $\mathbb{R}^n$ and let f be real valued and differentiable on C. Then f is *convex* [4] if

$$f(x) - f(y) \geq \langle \nabla f(y), x - y \rangle\ , \quad \forall\ x, y \in C$$

The function f is said to be *invex* [20] if there is a vector function $\eta : C \times C \to \mathbb{R}^n$

$$f(x) - f(y) \geq \langle \nabla f(y), \eta(x, y) \rangle\ , \quad \forall\ x, y \in C$$

Definition 2.4. Let X and Y be two normed vector spaces. A continuous linear transformation $A{:}X \to Y$ is said to be the *Fréchet*(Strong) derivative [8] of $f : X \to Y$ at x if for every $\epsilon > 0$, $\exists\ \delta > 0$ such that

$$\| f(x + h) - f(x) - Ah \|_Y \leq \epsilon \| h \|_X \qquad \forall h \text{ with } \| h \|_X \leq \delta$$

When the derivative exists it is denoted by $Df(x)$.

Proposition 2.1.[17] Let X be a vector space and Y be a normed space. Let S be a transformation mapping an open Set $D \subset X$ into an open set $E \subset Y$ and let P be a transformation mapping E into a normed space Z. Put $T = PS$ and suppose S is Frechet differentiable at $x \in D$ and P is Frĕchet differentiable at $y = S(x) \in E$. Then T is Frechet Differentiable at x and $DT(x) = DP(y)DS(x)$.

Remark:[4] It is to be noted that in $\mathbb{R}^n$, $Df(x) = \nabla f(x)$.

Definition 2.5. An ordering $\geq$ on a real vector space V is said to be *Archimedean* if $v \geq \theta_V$ whenever $u + \lambda v \geq \theta_V$ for some $u \in V$ and all $\lambda > 0$.

If $u \leq w$ and $u, w \in V$ then [u,w] will denote the set $\{v \in V : u \leq v \leq w\}$. Such a set is termed as *order interval*. A subset of V is *order bounded* if it is contained in some order interval.

Remark: The order relation $\geq$ in $\mathbb{R}^n$ is archimedean if $x + ny \geq \theta, n = 1, 2, 3, ...$ *implies* $y \geq \theta$. Most of the orderings that occur in practical problems are archimedean. Lexicographic orderings in sequence spaces are non-archimedean orderings.

Theorem 2.1(Banach-Alaoglu).[3] The closed unit ball B(0;1) in a Hilbert space H is weakly compact.

Theorem 2.2(Eberlin- Šmulian).[3] Let C be a subset of a Hilbert space H. Then C is weakly compact iff it is weakly sequentially compact.

Theorem 2.3(Generalized Weierstrass Theorem).[3] Let $C \subset H$ is a weakly compact set. Suppose $f : C \to \mathbb{R}$ is weakly sequentially lower semi-continuous. Then f is bounded below and has a minimizer on C.

3. INVEX PROGRAMMING PROBLEM(IP)

Definition 3.1. Let H_1 and H_2 be two real Hilbert spaces with some archimedean ordering " $\geq$ " and $X \subseteq H_1$ is an open invex set. The differentiable(Frechet) function $f : X \to H_2$ is *invex* if there exist a vector function $\eta : X \times X \to H_2$ and some $\mathbf{e} \in H_2$ with $\|\mathbf{e}\|_{H_2} = 1$ such that,

$$f(\mathbf{x}) - f(\mathbf{y}) \geq \langle Df(\mathbf{y}) , \eta(\mathbf{x}, \mathbf{y})\rangle \mathbf{e} \qquad \forall \mathbf{x}, \mathbf{y} \in X \tag{1}$$

Remark: It is to be noted that if H_1 and H_2 are taken as $\mathbb{R}^n$, then if we choose $\mathbf{e} = (1, 1, 1, ..., 1)$ and $\eta(x, y) = (x - y)$, f will become a convex function in $\mathbb{R}^n$. The norm in this case can be taken as $(n)^{-\frac{1}{2}}$-multiple of the usual euclidean norm.

Example 3.1. Let us consider the function $f : L^2[0, 1] \to L^2[0, 1]$ defined as,

$$(f(x))(t) = (x - sinx)(t) \ , \ x > 0, \ n \in \mathbb{N}$$

Clearly, f(x)is non-convex in nature. But it can be verified that f(x) is invex considering

$$\eta(x, y) = \frac{4sin\dfrac{x-y}{2}}{cos(x) - 1} \text{ whenever } x \neq 2n\pi \text{ and } \eta(x, y) = 0 \text{ elsewhere.}$$

Theorem 3.1. Let $f : X \to H_2$ be differentiable. Then f is invex iff every stationary point is a global minimizer.

Proof: Let f be invex and $Df(y) = 0$ for $y \in X$. Then clearly, $f(x) - f(y) \geq 0 \ \forall x \in X$. Therefore y is a global minimizer of f over X.

Now, let us assume that every stationary point is a global minimizer. If y is a stationary point, then (1) is obvious. Otherwise choose $\eta(x, y) = \dfrac{\|f(x) - f(y)\|}{\|D(f(y)\|^2} Df(y))$

and $\mathbf{e} = \dfrac{f(x) - f(y)}{\|f(x) - f(y)\|}$. It is to be noted that these are not the only choice for $\eta(\cdot,\cdot)$ and $\mathbf{e}$. □

Theorem 3.2. Let $f : X \to H_2$ and $g : X \to H_2$ be differentiable invex functions such that either $Df(y) = -\lambda Dg(y)$ for some $\lambda > 0$ and $f(x) - f(y) \geq -\lambda[g(x) - g(y)]$ or $Df(y) \neq -\lambda Dg(y)$ for any $\lambda > 0$. Then f and g are invex with respect to same $\eta(\cdot,\cdot)$ and $\mathbf{e}$.

Proof: Let us prove the theorem by contradiction. Let f and g be invex with respect to the same $\eta(\cdot,\cdot)$ and $\mathbf{e}$. Let us assume that there exist $x, y \in X$ and $\lambda > 0$ such that $Df(y) = -\lambda Dg(y)$ and $f(x) - f(y) < -\lambda[g(x) - g(y)]$. Now, since f and g are invex with respect to the same $\eta(\cdot,\cdot)$ and $\mathbf{e}$,

$$f(\mathbf{x}) - f(\mathbf{y}) \geq \langle Df(\mathbf{y}) \,,\, \eta(\mathbf{x},\mathbf{y})\rangle\mathbf{e}$$
$$g(\mathbf{x}) - g(\mathbf{y}) \geq \langle Dg(\mathbf{y}) \,,\, \eta(\mathbf{x},\mathbf{y})\rangle\mathbf{e}$$

Therefore,

$$\begin{aligned} f(\mathbf{x}) - f(\mathbf{y}) + \lambda(g(\mathbf{x}) - g(\mathbf{y})) &\geq \langle Df(\mathbf{y}) \,,\, \eta(\mathbf{x},\mathbf{y})\rangle\mathbf{e} + \lambda\langle Dg(\mathbf{y}) \,,\, \eta(\mathbf{x},\mathbf{y})\rangle\mathbf{e} \\ &= \langle Df(y) + \lambda Dg(y), \eta(x,y)\rangle\mathbf{e} = \theta \end{aligned}$$

Which contradicts the assumption. □

It is to be mentioned here that if $H_1 = H_2 = \mathbb{R}^n$, then using Gale's Theorem of the alternatives for linear inequalities, we can very easily prove that the above conditions are necessary as well.

Example 3.2. The functions $f(x) = -2x^2$ and $g(x) = log(x)$ are invex with respect to same $\eta(\cdot,\cdot)$ and $\mathbf{e}$. One of the several choice for $\eta(\cdot,\cdot)$ and $\mathbf{e}$ can be $x - y$ and 1 respectively.

Definition 3.2. Let H_1 and H_2 be two real Archimedean ordered seperable Hilbert spaces and I be an open invex set in H_1. Let $f, g, h : I \to H_2$ be differentiable (Frechet) invex functions with respect to same $\eta(\cdot,\cdot)$ and $\mathbf{e}$. Let us consider the following basic nonlinear programming problem

$$\begin{aligned} &Min \;\; f(x) \\ s.t. \;\; &g(x) \leq \theta_{H_2} \\ &h(x) = \theta_{H_2} \\ &x \geq \theta_{H_1} \end{aligned}$$

Let us refer the problem by Invex Programming Problem and denote the same by IP.

Example 3.3. Detection Filter Problem (Fortmann, Athans) [13] :

$$\begin{gathered} Min\{-\langle u, x\rangle : u \in L^2[0,T]\} \\ \text{s.t. } \langle u, s_t\rangle - \epsilon\langle u, s\rangle \leq 0 \quad \delta \leq |t| \leq T \\ -\langle u, s_t\rangle - \epsilon\langle u, s\rangle \leq 0 \quad \delta \leq |t| \leq T \\ \|u\| \leq 1 \end{gathered}$$

Where s is the signal function with the assumption that the energy of s equals to 1, i.e., $\|s\|^2 = 1$.

Before we proceed to the next section, let us have the following assumption for the rest of the discussion

Assumption 3.1. Let H be a Hilbert space with an Archimedean ordering " $\geq$ " , then $\theta \geq x \Rightarrow -cx \geq \theta$ for all $x \in H$ and for all scalar $c \geq 0$.

4. MAIN RESULT

The following theorem is a generalization of the very well known Fritz-John conditions. Under the assumption of invexity, the conditions are not only necessary but sufficient also. The proof of the necessity of the conditions is motivated by McShane [4]. In our discussion whenever we consider topology, we mean weak topology.

Theorem 4.1. $x^* \in I$ is a solution of IP iff there exist non-zero scalars λ, μ and ν such that

(i) $\lambda g(x^*) = \theta$

(ii)$\lambda Df(x^*) + \mu Dg(x^*) + \nu Dh(x^*) = \theta$

Proof: Let K be a strictly increasing differentiable real valued function defined on H_2 such that $K(x) > 0$ whenever $x > \theta$ and $K(x) = 0$ elsewhere. It is to be noted that $DK(x) > 0$ for $x > \theta$. Since g is continuous and I is open , there exist an $\epsilon_0 > 0$ such that $B(\theta, \epsilon_0) \subset I$ and for $g(x) \leq \theta$ for $x \in B(\theta, \epsilon_0)$. Now define a function

$$F(x, p) = ||f(x)|| + ||x||^2 + p\{K(g(x)) + ||h(x)||^2\}, \quad x \in I \text{ and } p \in \mathbb{Z}^+ \tag{2}$$

We assert that for each ϵ satisfying $0 < \epsilon < \epsilon_0$, there exist a positive integer $p(\epsilon)$ such that for x with $||x|| = \epsilon$, $F(x, p(\epsilon)) > \theta$. If not, then there would exist an ϵ' with $0 < \epsilon' < \epsilon_0$ such that for each positive integer p, there exist a vector x_p with $||x_p|| = \epsilon'$ and $F(x_p, p) \leq \theta$. Hence form (2),

$$||f(x)|| \leq -\{||x||^2 + p\{Kg(x_p) + ||h(x_p)||^2\}\} \tag{3}$$

Now since $||x_p|| = \epsilon'$ and since $S(0, \epsilon') = \{y : \ ||y|| = \epsilon'\}$ is weakly compact, then there exist sub-sequences, which we relabel as x_p and p, and a point x_0 with $||x_0|| = \epsilon'$ such that $x_p \mapsto x_0$. Since f, g and h are continuous, $f(x_p) \mapsto f(x_0)$; $g(x_p) \mapsto g(x_0)$; $h(x_p) \mapsto h(x_0)$. Therefore, dividing (3) by $-p$ and letting $p \to \infty$, we get, $K(g(x_0)) + ||h(x_0)||^2 = 0$. Hence, by definition of $K(\cdot, \cdot)$, $g(x_0) \leq \theta$ and $h(x_0) = \theta$. Thus x_0 is a feasible vector. Now, by a suitable affine transformation x^* can be assumed as θ and $f(x^*) = f(\theta) = 0$. Therefore, $f(x_0) \geq f(\theta) = 0$. Now from (3) $||f(x_p)|| \leq -(\epsilon')^2 < 0$, which is a contradiction. Hence, the assertion is true.

Again for each $\epsilon \in (0, \epsilon_0)$, the function $F(\cdot, p(\epsilon))$ is continuous on the closed ball $\overline{B(0, \epsilon)}$. Since $\overline{B(0, \epsilon)}$ is weakly compact, $F(\cdot, p(\epsilon))$ attains its minimum on $\overline{B(0, \epsilon)}$ at an interior point x_ϵ of $\overline{B(0, \epsilon)}$. Hence

$$DF(x_\epsilon, p(\epsilon)) = 0 \tag{4}$$

Now let us assume that

$$
\begin{aligned}
L(\epsilon) &= 1 + (p(\epsilon)DK(g(x_\epsilon)))^2 + (p(\epsilon)D(\|h(x_\epsilon)\|))^2 \\
\lambda(\epsilon) &= \frac{D(\|f(x)\|)}{\sqrt{L(\epsilon)}} \\
\mu(\epsilon) &= \begin{cases} \dfrac{p(\epsilon)DK(g(x_\epsilon))}{\sqrt{L(\epsilon)}} & \textit{if } \ g(\theta) = \theta \\ 0 & \textit{else} \end{cases} \\
\nu(\epsilon) &= \frac{p(\epsilon)D(\|h(x_\epsilon\|)}{\sqrt{L(\epsilon)}}
\end{aligned} \tag{5}
$$

It is to be noted that $\lambda(\epsilon) \geq 0$,and $\mu(\epsilon), \nu(\epsilon) \geq \theta$.
Now from (2), (4), (5) we get

$$
\lambda(\epsilon)Df(x_\epsilon) + \frac{D(\|x_\epsilon\|^2)}{\sqrt{L(\epsilon)}} + \mu(\epsilon)Dg(x_\epsilon) + \nu(\epsilon)Dh(x_\epsilon) = 0 \tag{6}
$$

Let $\epsilon \to 0$ through a sequence of values ϵ_k. Then, since $\|x_\epsilon\| < \epsilon$, we have

$$
x_{\epsilon_k} \to \theta, \ \lambda(\epsilon_k) \to \lambda, \ \mu(\epsilon_k) \to \mu, \ \nu(\epsilon_k) \to \nu \tag{7}
$$

Therefore, from (6) and (7) we get $\lambda Df(\theta) + \mu Dg(\theta) + \nu Dh(\theta) = \theta$, and from the definition of μ, $\mu g(\theta) = \theta$. This proves the necessity of the conditions.
Let us now consider the sufficiency of the conditions.

$$
\begin{aligned}
f(x) - f(x^*) &\geq \langle Df(x^*), \eta(x, x^*)\rangle e \\
&= -\langle \mu Dg(x^*) + \nu Dh(x^*), \eta(x, x^*\rangle e \\
&= -\{\mu\langle Dg(x^*), \eta(x, x^*\rangle e + \nu\langle Dh(x^*), \eta(x, x^*\rangle e\} \\
&\geq -\{\mu(g(x) - g(x^*)) + \nu(h(x) - h(x^*))\} \\
&= -\mu g(x) \\
&\geq \theta
\end{aligned}
$$

which proves the sufficiency of the conditions. □

5. CONCLUSION

It is quite obvious that using any constraint qualification to assure the positivity of λ, we can obtain a generalization of the very popular Karush-Kuhn-Tucker conditions from Theorem 4.1. Thus, the theorem can be extremely useful in solving a wide class of optimization problems in infinite dimensional Hilbert spaces.

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