International Review of Pure and Applied Mathematics
(January-June) 2010, Volume 6, No. 1, pp. 23–29

Characterization of Transportation Problem in Infinite Dimensional Spaces

Sandip Chatterjee & R. N. Mukherjee

Abstract: In this paper the authors have considered an infinite dimensional transportation problem and showed how to find out the solution of those problems.

2000 Mathematics Subject Classification: 49F, 93E, 46N, 49J, 49N.

Keywords: Conjugate Spaces, γ-algebra, σ-finite measure, Optimal Solution Spaces, Moore Penrose Inverse.

1. Introduction

Ben-Israel and Charnes [2] first determined an explicit solution of a Linear Programming Problem using the generalized inverse of the coefficient matrix. Those problems were further extended in [17 & 18]. Also some illustrative methods were shown by the authors[17 & 18]. Sivakumar and Kulkarni [11, 12 & 13] extended the results of [2, 17 & 18] in infinite dimensional spaces.

Further K. C. Sivakumara and J. Mercy Swarna [15] considered LPP in infinite dimensional space. They have characterized the solution of those problems in corresponding space by functional analytic technique.

Due to the particular nature of the transportation problem the authors in this paper have developed a uniform theory for solving transportation problems in infinite dimensional space.

Moreover they have shown that the results obtained are the same as in the finite dimensional case which exist in standard literature. An example has been sited to illustrate the theory.

Let $(X_1 \times X_1, Y_1 \times Y_1)$ and (X_2, Y_2) be two dual pairs of real Banach Spaces, where X_1 and X_2 are real Banach Spaces and Y_1, Y_2 are conjugate spaces respectively, i.e. $Y_1 = X_1^*$ and $Y_2 = X_2^*$. Let $\propto = (x, y) \in X_1 \times X_1$

Let $A_x, A_y : X_1 \times X_1 \rightarrow X_2$ be two linear napping such that

$$A_x(\mu) = x \quad \text{and} \quad A_y(\propto) = y.$$

Let $A : X_1 \times X_1 \rightarrow X_2 \times X_2$ be a linear operator such that $A = (A_x, A_y)$. Let $\theta \in Y_1 \times Y_1$. Then consider the following Linear Programming Problem:

Minimize $\theta(\propto)$

Subject to $A(\propto) = (a, b)$, where $(a, b) \in X_2 \times X_2$.

Such problems will be called as Infinite Primal Transportation Problem (IPTP). Clearly the dual of this problem will be called Infinite Dual Transportation Problem (IDTP) and will be as follows:

Maximize $w(a) + v(b)$

Subject to $A'(w, v) \leq \theta$.

Where $w, v \in Y_2$.

In this paper we will mainly discuss the characterization of these problems in Infinite Dimensional Banach Spaces. Here we take approach similar to [2]. In the next section we present the preliminaries that are required in the rest of the paper.

2. Preliminaries & Notations

Definition 2.1: Let X be a real vector space. Then X is called partially ordered vector space if X has a partial order '≤' defined on it satisfying the following: for $x, y \in X$ and $x \leq y$, $x + u \leq y + u \forall\ u \in X$ and $\propto x \leq \propto y\ \forall$ real scalar $\propto\ \geq 0$.

Definition 2.2: Let X be a partially ordered real vector space. Then the subset $C := \{x \in X \mid x \geq 0\}$ is called the positive cone of X. C will be called a strictly positive cone if $x \leq y$ and $y \leq x$ imply $x = y$ for every $x, y \in X$.

Definition 2.3: A vector space is said to be partially ordered Banach Space if it is a partially ordered vector space and also a Banach Space with respect to a suitable norm.

Definition 2.4: Let $X = X_1 \times X_1$ be a real Banach Space and X_2 be a partially ordered Banach Space with P_2 as the positive cone. Let $A : X \to X_2$ be linear such that $A = (A_x, A_y)$. Let $\theta \in Y_1 \times Y_1$ and $a, b \in X_2$. Consider the problem called Infinite Primal Transportation Problem denoted as IPTP $(a, b, \theta, \mathrm{A})$

Min $\theta(\propto)$

Subject to $A(\propto) = (a, b)$.

A vector $\propto^* \in X$ is said to be feasible if $A(\propto^*) = (a, b)$. A feasible vector $\propto^*$ is said to be optimal if $\theta(\propto^*) \leq \theta(\propto)$ for every feasible vector $\propto$.

Remark: It is to be noted that the mapping A is actually the identity mapping on X.

$$\text{Let } (x, y) \in X. \text{ Now } A(x, y) = (\mathrm{A}_x, \mathrm{A}_y)(x, y)$$
$$= (A_x(x, y), A_y(x, y)) = (x, y)$$

which implies $A \equiv I$.

Definition 2.5: Let X be a partially ordered real Banach space. Let I denote the identity map on X. We say that X is an Optimal Solution space if IPTP (a, b, θ, I) has an optimal solution for all $a, b \in X$ and $\theta \in X^*$.

Definition 2.6: A class of subsets of an arbitrary space X is said to be a σ-algebra if X belongs to the class and the class is closed under the formation of countable unions and of complements.

Definition 2.7: A measure μ on a class of sets $\mathcal{R}$ is σ-finite if, for every set $E \in R$, we have $E = \bigcup_{n=1}^{\infty} E_n$ for some sequence $\{E_n\}$ such that $E_n \in \mathcal{R}$ and $\mu(E_n) < \infty$ for each n.

3. Some Results on Optimal Solution Space

Theorem 3.1: Let m be a δ-finite positive measure on a δ-algebra $\mathcal{M}$ in a non-empty set Y and $L^p(Y, \mu)$, $1 \leq p < \infty$ be a space of (equivalent classes) measurable p-integrable functions on Y. Then $L^p(Y, \mu)$ is an Optimal Solution Space.

Proof: Let $X = L^p(Y, \mu)$, $1 \leq p < \infty$ and $C := \{f \in X : f \geq 0\ a.e.\ (\mu)\}$. Then X is partially ordered Banach Space with C as the positive cone. Let $\theta \in X^*$. Then there exists a unique $h \in L^q(Y, \mu)$, where q is the conjugate exponent of p, (by Reisz Representation Theorem) such that $\forall f \in X$

$$\theta(f) = \int_Y hf d\mu$$

Let $a, b \in X$ with $a \leq b$ and IPTP (a, b, θ, A) be feasible. Define $Y_+ := \{y \in Y : h(y) \geq 0\}$; $Y_- := \{y \in Y : h(y) \geq 0\}$ and $\eta = b\chi_{Y_-} + a\chi_{Y+}$, where χ_S denote the characteristic function of a set S. It follows then that χ is measurable.

Further $\int |\eta|^p d\mu \leq \|b\|^p + \|a\|^p$.

This shows that $\eta \in X$.

For any $u \in X$ satisfying $a \leq u \leq b$ a.e.(μ) we have

$$\begin{aligned} \theta(\eta - u) &= \int_Y h(\eta - u) d\mu \\ &= \int_{Y_+} h(\eta - u) d\mu + \int_{Y_-} h(\eta - u) d\mu \\ &\leq 0 \\ &\Rightarrow \theta(\eta) \leq \theta(v) \end{aligned}$$

This η is optimal for the problem IPTP (a, b, θ, A), i.e. $L^p(Y, \mu)$ is an Optimal Solution Space.

Theorem 3.2: Let μ be a σ-finite positive measure on a σ-algebra $\mathcal{M}$ in a non-empty set Y and $L^p(Y, \mu)$, $1 \le p < \infty$ be the space of (equivalent classes) measurable p-integrable functions on Y. Then $L^p(Y, \mu) \times L^p(Y, \mu)$ is an Optional Solution space.

Proof: Let $X = L^p(Y, \mu) \times L^p(Y, \mu)$

$$C := \{(f, g) \in X \times X : f \le 0 \quad \text{and} \quad g \ge 0 \ \ a.e.in\,(\mu)\}$$

Then X is a partially ordered Banach Space with C as a positive cone. Let $\phi = (\theta, \xi)\, \varepsilon X^*$. Then there exists a unique $H = (h_1, k_1)\, \varepsilon\, L^p(Y, \mu) \times L^p(Y, \mu)$ such that

$$\phi(f, g) = (\theta, \xi)\,(f, g) \int_{Y \times Y} h_1 fd\,(\mu \times \mu) \int_{Y \times Y} k_1 gd\,(\mu \times \mu)$$

Let $a, b \in X$ and IPTP (a, b, ϕ, A) be fasible.

Define
$$(Y \times Y)_+ := \{(y_1, y_2) : h_1(y_1), K_1(y_2) \le 0\};$$
$$(Y \times Y)_- := \{(y_1, y_2) : h_1(y_1), K_1(y_2) < 0\}$$

and
$$\gamma = b\chi_{(Y \times Y)_-} + b\chi_{(Y \times Y)+}$$

where χ_S denote the characteristic function of the set S. It is evident that γ is measurable.

Further

$$\int_{Y \times Y} 1\gamma 1^p\, d\,(\mu \times \mu) = \int_{Y \times Y} 1 b\gamma_{(Y \times Y)_-} + a\chi_{(Y \times Y)_+}|^p\, d\,(\mu \times \mu)$$
$$= \| b \|^p + \| a \|^p$$

$\therefore$ $\gamma \varepsilon X$. For any feasible solution $u \in X$ s.t. $a \le u \le b$

We have

$$\phi\,(\gamma - u) = \int_{Y \times Y} h_1(\gamma - u)\, d\,(\mu \times \mu) \cdot \int_{Y \times Y} K_1(\gamma - u)\, d\,(\mu \times \mu)$$
$$\le 0$$

$\therefore$ $\phi\,(\gamma) \le \phi\,(x)$.

$\therefore$ γ is optimal for the problem IPTP (a, b, θ, A) i.e. $L^p(Y, \mu) \times L^p(Y, \mu)$ is an optional solution space.

Remark: It can be proved that the product of an arbitrary no. of Optimal Solution Spaces is an Optimal Solution Space.

4. An Algorithm for Solving the IPTP

Step I: Verify whether the operator in the constraint admits any inverse or not. If it is not having any inverse then determine the Moore-Penrose inverse [18] of it.

Step II: Standardize the infinite transportation problem i,e. we have to operate the inverse (or Moore-Penrose inverse) on the constraint to convert it into the form of IPTP.

Step III: Compare both the sides of the constraint to get the optimal solution.

Remark: It is to be noted that IPTP admits unique optimal solution because of the existence of identity operator in the constraints. Now a transportation problem in infinite dimensional space may be considered as infinite number of sources with some available resources and infinite number of destinations with some demands. Then the condition that an IPTP is solvable is that the infinite series of the availabilities and the infinite series of the demands must converge. Now since the convergence is unique the problem will have a unique optimal solution.

The theory is illustrated by the following example:

Example 4.1: Let $X = L^2[0, 1]$. Define $A = X \times X \to X \times X$ by

$$A(x_1(s), x_2(s)) = \left(s\int_0^t t^2 x_1(t)\,dt + s^2 \int_0^t t x_1(t)\,dt, u\int_0^t t^2 x_2(t)\,dt + u^2 \int_0^t t^2 x_2(t)\,dt \right)$$

Let $x_2(s) = \mathbf{0}$.Which implies that

$$A(x_1(s), \mathbf{0}) = \left(s\int_0^t t^2 x_1(t)\,dt + s^2 \int_0^t t x_1(t)\,dt, o \right) \quad s, t\,\varepsilon\,[0, 1]$$

Let $b(s) = \dfrac{1}{12}s^2 + \dfrac{1}{20}s$ and $\theta(s) = s^2$

Consider the following IPTP:

$$\text{Minimize} \left(\int_0^1 t^2 x_1(t)\,dt, 0 \right)$$

Subject to

$$A(x_1(s), 0) = \left(\frac{1}{12}s^2 + \frac{1}{20}s, 0 \right)$$

$$x_1(s) \geq \mathbf{0},\ t, s\,\varepsilon\,[0, 1]$$

It follows that the Moore-Penrose inverse of A is given by $A^{+} = DA$ where

$$D = \left(\frac{1}{240} \begin{pmatrix} 31 & -24 \\ -40 & 31 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right)$$

Now it can be verified that the optimal value is $\left(\frac{1}{20}, 0 \right)$

Note: Assignment problem in infinite dimensional spaces can be considered as a particular case if one consider the spaces under consideration are of the same cardinality (i,e. they are isomorphic).

REFERENCES

[1] Anderson E. J. and Nash P., *Linear Programming in Infinite Dimensional Spaces*, John Willey and Sons, (1987).

[2] Ben-Israel A. and Charnes A., An Explicit Solution of a Special Class of Linear Programming Problems, *Operation Research*, **16**, (1968), 1166–1175.

[3] Barra G. D., *Measure Theory and Integration*, Willey Eastern Limited, (1991).

[4] Charnes A., Cooper W. W. and Kortanek K. O., On Representations of Semi-Infinite Programs which have no Duality Gaps, *Management Sci.*, **12**, (1965), 113–21.

[5] Charnyi V. I., Decomposition of Linear Functional Problems I: Linear Programming in Hilbert Spaces, *Aut. & Rem. Control*, **39**, (1978), 874-82.

[6] Grinold R. C., Continuous Programming, Part One: Linear Objectives, *J. Math. Anal. Appl.*, **28**, (1969a), 32–51.

[7] Groetch C. W., Generalized Inverses of Operators: Representation and Approximation, *Marcel Dekkar*, (1977).

[8] Hoffman K. H. and Klostermair A., A Semi-Infinite Linear Programming Procedure and Applications to Approximation Problems in Optimal Control, in G.G. Lorentz, (Ed.), *Approximation Theory II*, Academic Press, New York, (1976).

[9] Holmes R. B., Geometric Functional Analysis and its Applications, *Springer-Verlag*, New York, (1975).

[10] Hanson M. A., Duality for a Class of Continuous Linear Programming Problems, *SIAM. J. Appl. Math.*, **16**, (1968), 318–??.

[11] Kulkarni S. H. and Sivakumar K. C., Explicit Solutions of a Special Class of Linear Economic Models, *Ind. J. Pure & Appl. Math.*, **26**(3), (1995), 217–223.

[12] Kulkarni S. H. and Sivakumar K. C., Explicit Solutions of a Special Class of Linear Programming Problems in Banach Spaces, *Acta. Sci. Math., (Szeged)*, **62**, (1996), 457–465.

[13] Kulkarni S. H. and Sivakumar K. C., Application of Generalized Inverses to Interval Linear Programmings in Hilbert Spaces, *Numer, Funct. Anal & Optimiz.*, **16**(7&8), (1995), 965–973.

[14] Kreyszig E., Introductory Functional Analysis with Applications, J. Willey & Sons, Inc., (1978).

[15] Sivakumar K. C. and Swarna J. Mercy, Explicit Solvability of Dual Pairs of Infinite Linear Programs, *OPSEARCH*, **42**, (2005), 288–96.

[16] Schechter M., Duality in Continuous Linear Programming, *J. Math. Anal. Appl.*, **37**, (1972a), 130-41.

[17] Zlobec S. and Ben-Israel A., On Explicit Solutions of Interval Linear Programs, *Israel J. Math*, **8**, (1970), 12–22.

[18] Zlobec S. and Ben-Israel A., Explicit Solutions of Interval Linear Programs, *Operations Research*, **21**, (1973), 390–393.

Sandip Chatterjee
Dept. of Mathematics, Heritage Institute of Technology,
Kolkata-700109, (W.B), India
E-mail: functionals@gmail.com

R. N. Mukherjee
Dept. of Mathematics, The University of Burdwan,
Burdwan-713104, (W.B), India
E-mail: rnm_bu_math@yahoo.co.in