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Free vibration analysis of axially functionally graded linearly taper beam on elastic foundation

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Abstract. In the present study non-linear free vibration analysis is performed on a linearly tapered Axially Functionally Graded (AFG) beam resting on an elastic foundation with different boundary conditions. Firstly the static problem is carried out through an iterative scheme using a relaxation parameter and later on the subsequent dynamic problem is solved as a standard eigen value problem. Minimum potential energy principle is used for the formulation of the static problem and for the dynamic problem Hamilton's principle is utilized. The free vibrational frequencies are tabulated for different taper parameter and different foundation stiffness. The dynamic behaviour of the system is presented in the form of backbone curves in dimensionless frequency-amplitude plane.

1. Introduction

Multilayer composite materials find large-scale utilisation in aerospace, civil, mechanical engineering applications, as well as automotive and nuclear industries due to their outstanding behaviour such as high ratio of stiffness and strength to weight, low maintenance cost etc. But contemporary laminated composite materials exhibit a mismatch of mechanical properties at the layer interface due to bonding of two discrete materials. As a result there is a chance of stress concentration and residual stresses at the interface [1], which can lead to damage in the form of delamination, matrix cracking and adhesive bond separation. Functionally graded materials (FGMs) are free of these disadvantages as material properties are obtained as a function of spatial position resulting in a continuous variation from one surface to another. The continuous gradation is achieved by combination of two or more constituent materials, mixed continuously and functionally according to a given volume fraction. The variation of material properties in functionally graded (FG) beams may be oriented in transverse (thickness) direction or longitudinal/axial (length) direction or both.

Various engineering structures are often represented as slender beams resting on elastic foundation. These structures can be modelled as beams supported by a series of springs, whose stiffness characterises the property of the foundation. Similarly, non-uniform beams, due to their favourable strength and mass distribution, find wide application in turbine blades, ship propellers, robot arms, helicopter rotor blades, space and marine structures etc. [2]

An exhaustive literature review of the relevant domain reveals that majority of the studies are concentrated on free vibration analysis of uniform beams on elastic foundation with material property variation along the depth of the beam. Kodali *et al* implemented displacement field based on higher order shear deformation theory to study the static behavior of functionally graded metal–ceramic (FGM) beams under ambient temperature. The material properties of the functionally graded materials were assumed to be graded according to power law in the thickness direction [3]. Yaghoobi *et al* carried out the bending analysis of simply supported FG beam [4]. Hajilar *et al* studied the free vibration and stability analysis of axially functionally graded tapered Timoshenko beams through a finite element approach. The effects of taper ratio, elastic constraint, attached mass and material non-homogeneity on the natural frequencies and critical buckling load were investigated [5]. Rajasekaran *et al* investigated the free vibration and stability of axially functionally graded tapered Euler–Bernoulli beams through solving the governing differential equations of motion [6]. Kumar *et al* [7] and Sarkar *et al* [8] carried out free vibration analysis on axially functionally graded (AFG) tapered slender beams under different boundary conditions. Kanani *et al* [9] investigated the large amplitude free and forced vibration of FG beam resting on nonlinear elastic foundation containing shearing layer and cubic nonlinearity. Civalek *et al* [10] investigated the bending response of non-homogenous microbeams embedded in an elastic medium based on modified strain gradient elasticity theory in conjunctions with various beam theories. Niknam *et al* [11] made an attempt to obtain a closed form solution for both natural frequency and buckling load of non-local FG beams resting on nonlinear elastic foundation.

Literature review reveals that the field of free vibration study of depth-wise functionally graded beams is explored quite comprehensively, while analysis of axially functionally graded (AFG) beams has gained prominence recently. However, the domain of AFG beam on elastic foundation, which incorporates a higher level of complexity, remains largely unexplored. Hence, the present study is taken up with the objective of analysing the large amplitude free vibration behaviour of axially functionally graded (AFG) linearly tapered beams on elastic foundation. Variation of material properties (elastic modulus and density) along the length of the beam is considered according to specified functions. The large amplitude free vibration behaviour is presented as backbone curves in non-dimensional amplitude-frequency plane, where, variation of natural frequency with the maximum amplitude of deflection yields the backbone curve of the system. Effect of variation of system geometry (taper parameter) on the dynamic behaviour is also studied.

2. Mathematical Formulation

For the present analysis an axially functionally graded non-uniform beam of length L, breadth b and variable thickness $t(x)$ is considered (Figure 1). The beam is considered to be resting on an elastic foundation, which is idealised as a series of linear springs of stiffness K, attached to the bottom surface of the beam. The modulus of elasticity, $E(x)$, and the mass density, $\rho(x)$ of the beam vary along the axial direction according to the following functions, $E(\xi) = E_0(1+\xi), \rho(\xi) = \rho_0(1+\xi+\xi^2)$, where, E_0 and ρ_0 are the elastic modulus and density at the left hand edge of the beam (Figure 1). For the mathematical formulation, normalized coordinate ($\xi = x/L$) is taken into account. It is assumed that the cross-sectional dimensions are sufficiently smaller than the length of the beam to neglect the effect of shear deformation and rotary inertia.

Geometrical nonlinearity is taken into consideration by incorporating nonlinear strain-displacement relations. The problem is formulated in such a way that the static solution of the system subjected to transverse uniformly distributed load is obtained first, followed by standard eigenvalue problem on the basis of known static displacement field. For both static and dynamic analysis, formulation is carried out on the basis of variational form of energy principle. As geometric nonlinearity is incorporated in the formulation, both bending and stretching effects appear in the expression of strain energy of the system. Total strain energy in the system is given by, $U = U_b + U_m + U_f$, where, U_b and U_m are strain energy stored in the beam due to bending and stretching, respectively, while, U_f is strain energy of the total strain energy stored in the springs. The final expression of the total strain energy is obtained as,

$$U = \left(\frac{1}{2}\right)\int_0^L Kw^2 dx + \left(\frac{1}{2}\right)\int_0^L (d^2w/dx^2)^2 E(x)I(x)dx + \left(\frac{1}{2}\right)\int_0^L [(du/dx)^2 + (1/4)(dw/dx)^4 + (dw/dx)^2(du/dx)]\, E(x)A(x)dx \quad (1)$$

The work potential due to external loading is given as, $V = \int_0^L P(x)wdx$ (2)

2.1. Static analysis

The energy functionals of the previous segment are expressible in terms of assumed displacement fields, w and u, which are considered as linear combinations of unknown coefficients (c_i) and orthogonal admissible functions (ϕ and ψ). $w(\xi) = \sum_{i=1}^{nw} c_i\phi_i(\xi), u(\xi) = \sum_{i=1+nw}^{nw+nu} c_i\psi_{i-nw}(\xi)$ (3)

Applying the principle of minimum total potential energy, $\delta(U - V) = 0$ and substituting the expressions for U and V, as well as the assumed displacement fields the governing set of equations are derived in matrix form. $[K_s][c] = \{f\}$ (4)

The form of stiffness matrix and load vector are given by, $[K_s] = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$ and $\{f\} = \{f_{11} \quad f_{12}\}^T$. The elements of the stiffness matrix and load vector are,

$$\begin{aligned}[K_{11}] = {} & (1/L^3)\sum_{j=1}^{nw}\sum_{i=1}^{nw}\int_0^1 (d^2\phi_i/d\xi^2)(d^2\phi_j/d\xi^2)\,E(\xi)I(\xi)d\xi \\ & + (1/2L^3)\sum_{j=1}^{nw}\sum_{i=1}^{nw}\int_0^1 \left(\sum_{i=1}^{nw} c_i(d\phi_i/d\xi)\right)^2 (d\phi_i/d\xi)(d\phi_j/d\xi)E(\xi)A(\xi)d\xi \\ & + (1/L\ \ )\sum_{j=1}^{nw}\sum_{i=1}^{nw}\int_0^1 \left(\sum_{i=1+nw}^{nw+nu} c_i(d\psi_{i-nw}/d\xi)\right)^2 (d\phi_i/d\xi)(d\phi_j/d\xi)E(\xi)A(\xi)d\xi \\ & + KL\sum_{j=1}^{nw}\sum_{i=1}^{nw}\int_0^1 p(\xi)\,\phi_i\phi_j d\xi\end{aligned}$$

$$[K_{12}] = 0$$

$$[K_{21}] = (1/2L^2)\sum_{j=nw+1}^{nw+nu}\sum_{i=1}^{nw}\int_0^1 \left(\sum_{i=1}^{nw} c_i(d\phi_i/d\xi)\right)^2 (d\phi_i/d\xi)(d\psi_{i-nw}/d\xi)E(\xi)A(\xi)d\xi$$

$$[K_{22}] = (1/L)\sum_{j=nw+1}^{nw+nu}\sum_{i=nw+1}^{nw+nu}\int_0^1 (d\psi_{i-nw}/d\xi)(d\psi_{i-nw}/d\xi)E(\xi)A(\xi)d\xi$$

$$\{f_{11}\} = L\sum_{j=1}^{nw}\int_0^1 p(\xi)\,\phi_i d\xi$$

$$\{f_{12}\} = 0$$

2.2. Dynamic analysis

The free vibration problem is formulated on the basis of Hamilton's principle, which is mathematically expressed as, $\delta\left(\int_{\tau_1}^{\tau_2}(T - U)d\tau\right) = 0$ (5)

Here, T represents the kinetic energy of the system. $T = \frac{1}{2}\int_0^L \{\dot{w}^2 + \dot{u}^2\}\rho(x)A(x)dx$ (6)

It should be mentioned here that the springs are taken to be mass less and hence do not contribute towards the total kinetic energy of the system. Approximate dynamic displacement fields w and u are assumed as linear combination of unknown coefficients (d_i) and orthogonal admissible functions (ϕ and ψ),

$$w(\xi,\tau) = \sum_{i=1}^{nw} d_i\varphi_i(\xi)e^{j\omega\tau} \quad u(\xi,\tau) = \sum_{i=nw+1}^{nw+nu} d_i\psi_i(\xi)e^{j\omega\tau} \tag{7}$$

where, ω is the natural frequency of the system and d_i represents a new set of unknown coefficients that represents the eigenvectors in matrix form. Substituting Equations (1) and (5) along with the dynamic displacement fields in Equation (7) gives the governing set of equations for the beam in the following form,

$$-\omega^2[M]\{d\} + [K\{d\}] = 0 \tag{8}$$

Here, [M] is mass matrix, which has the following form and elements: $[M] = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$

$$M_{11} = L\sum_{j=1}^{nw}\sum_{i=1}^{nw}\int_0^1 \varphi_i\varphi_j\ \rho(\xi)A(\xi)d\xi, \qquad M_{12} = 0,\ M_{21} = 0,$$

$$M_{22} = L\sum_{j=nw+1}^{nw+nu}\sum_{i=nw+1}^{nw+nu}\int_0^1 \psi_{i-nw}\psi_{i-nw}\,\rho(\xi)A(\xi)d\xi$$

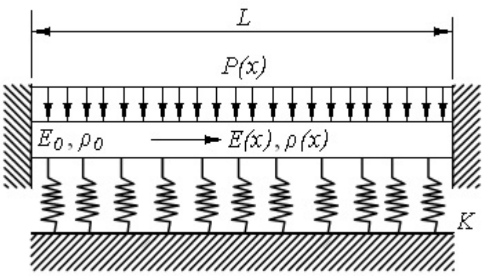


Figure 1. Beam on Elastic Foundation.

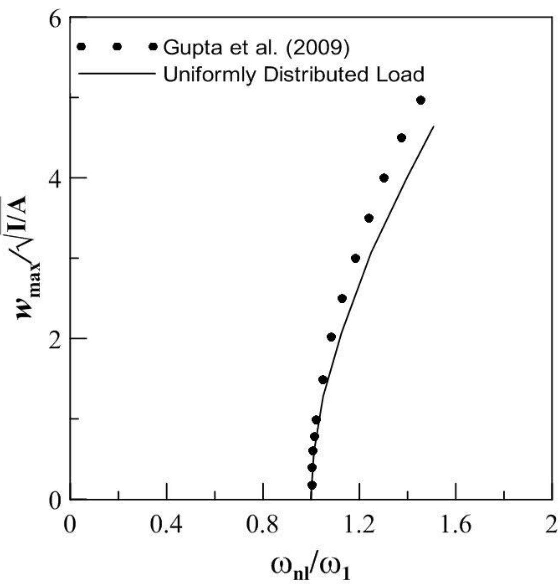


Figure 2. Comparison of backbone curves for fundamental mode of a clamped-clamped homogeneous uniform beam.

3. Results and Discussions

The objective of the present study is to investigate the large amplitude dynamic behaviour of axially functionally graded linear taper beams supported by elastic foundation. Variation of the loaded natural frequencies with change in taper parameter and stiffness of the foundation is also investigated. Here, the beam under consideration is taken to have uniform width and variation of thickness for the linear taper beam is given by the following equation, $t(\xi) = t_0(1 - \alpha\xi)$, where, t_0 is the thickness of the beam at the left hand edge and α is the taper parameter. For the present analysis, four different values of the taper parameter is considered as 0, 0.2, 0.4 and 0.6, respectively. Five values of the stiffness parameter (K) are taken as 0 N/m, 1000 N/m, 10000 N/m, 50000 N/m and 100000 N/m. It should be pointed out that the 0 stiffness condition refers to a situation where the springs are absent, i.e., the beam is not supported on elastic foundation.

In the present analysis, it is considered that the AFG taper beam on elastic foundation is subjected to uniformly distributed load for CC, CF, SS, SF flexural boundary condition. The in-plane displacements at the boundaries are assumed as zero. The displacement fields corresponding to these boundary conditions are generated as follows. At first, the start functions for the different situations are selected satisfying the flexural boundary conditions for transverse displacement (w) and membrane boundary condition for in-plane displacement (u). These start functions are tabulated in Table 1.

Table 1: Start functions for assume displacement field (w, u)

Flexural Boundary Condition	$\boldsymbol{\phi_1(\xi)}$
CC	$\{\xi(1-\xi)\}^2$
CF	$\xi^2(\xi^2-4\xi+6)$
SS	$\sin(\pi\xi)$
CS	$\xi^2(2\xi^2-5\xi+3)$
In-plane Boundary Condition	$\psi_1(\xi)$
Immovable	$\xi(1-\xi)$

Table 2: Values of natural frequencies for 1st and 2nd mode (ω_1 and ω_2) for different combinations of taper parameter and spring stiffness.

Taper Parameter(α)	**Stiffness (K) N/m**	**Material**							
		CC		**CF**		**SS**		**CS**	
		ω_1	ω_2	ω_1	ω_2	ω_1	ω_2	ω_1	ω_2
0	0	24.23	66.91	02.88	22.10	10.73	43.21	15.71	53.46
	1000	24.61	67.05	04.62	22.49	11.55	43.43	16.25	53.64
	10000	27.78	68.31	11.75	25.68	17.22	45.38	20.41	55.18
	50000	38.78	73.66	25.37	36.75	31.75	53.21	33.00	61.60
	100000	49.08	79.88	35.49	47.07	43.21	61.97	43.75	68.85
0.2	0	21.56	59.73	02.98	20.65	09.66	38.64	14.49	48.18
	1000	22.03	59.90	04.95	21.11	10.65	38.90	15.14	48.39
	10000	25.83	61.43	12.80	24.85	17.17	41.26	20.01	50.25
	50000	38.35	67.84	22.77	37.24	32.94	50.50	33.97	57.83
	100000	49.65	75.13	38.92	48.49	45.20	60.25	45.62	66.16
0.4	0	18.73	52.09	03.11	19.10	08.50	33.83	13.16	42.57
	1000	19.33	52.31	05.39	19.66	09.75	34.17	13.96	42.83
	10000	24.03	54.25	14.25	24.16	17.31	37.10	19.78	45.17
	50000	38.46	62.18	31.13	38.25	34.61	48.11	35.47	54.37
	100000	50.97	70.88	43.76	50.63	47.93	59.18	48.28	64.08
0.6	0	15.66	43.73	03.31	17.42	07.17	28.64	11.63	36.43
	1000	16.47	44.04	06.07	18.16	08.83	29.11	12.70	36.80
	10000	22.52	46.68	16.43	23.84	17.78	33.01	19.88	39.93
	50000	39.43	56.99	36.15	40.34	37.03	46.58	37.87	51.60
	100000	53.50	67.71	51.00	54.31	51.81	59.35	52.27	63.23

Gram-Schmidt orthogonalization scheme is used to generate the higher order functions and the number of functions is taken as 8 for each displacement. The same method can be utilized to handle non-classical boundary conditions like elastically restrained ends. However, to limit the volume of the present paper, only results pertaining to CC, CF, SS, SF boundaries are furnished on a beam on elastic foundation. A material model where the elastic modulus and density vary along the axial direction is considered and the expressions for these two parameters as function of the normalized axial coordinate are given as follows, $E(\xi)=E_0(1+\xi), \rho(\xi)=\rho_0(1+\xi+\xi^2)$. However, the present formulation and solution methodology is such that any other material model expressible as a mathematical function of the normalised axial coordinate can be handled.

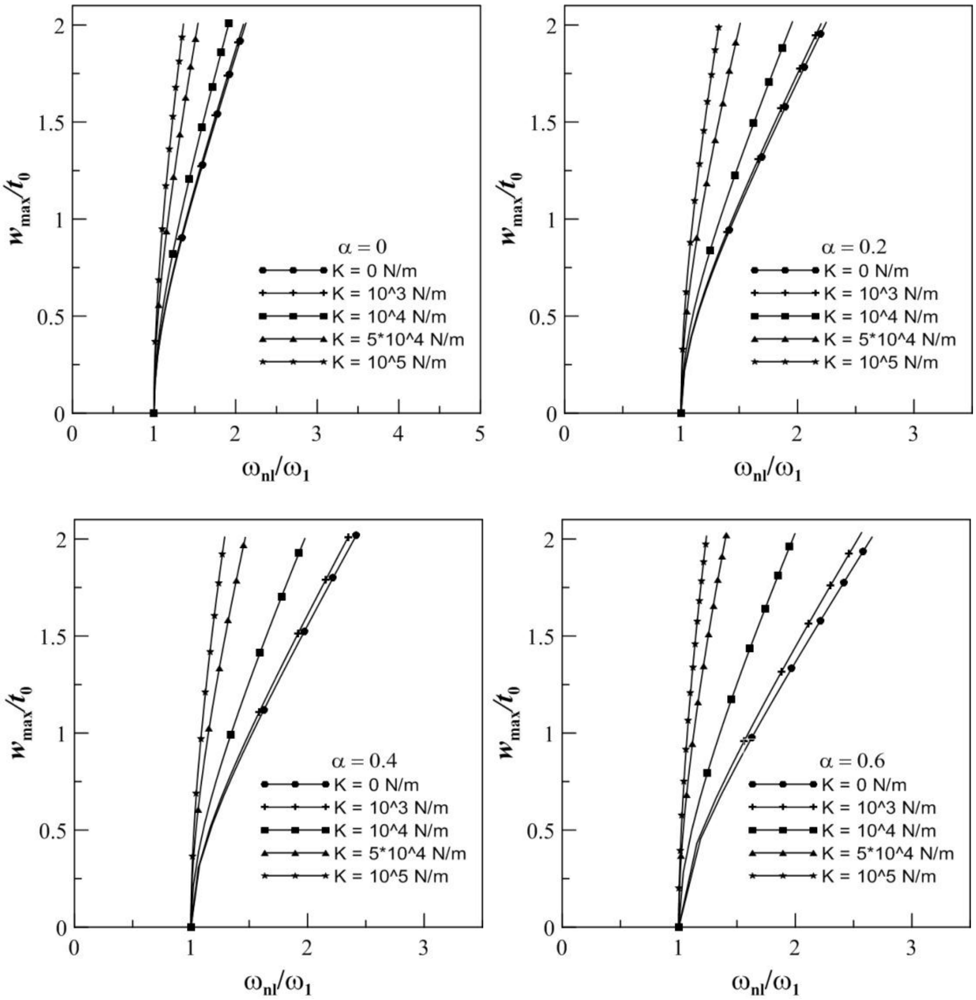
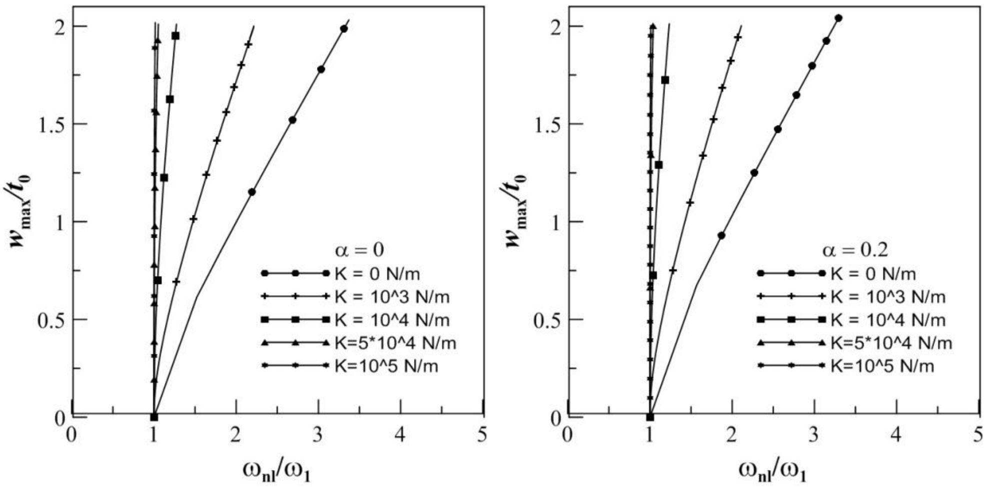


Figure 3. Backbone curve of AFG linear taper beam on elastic foundation with CC boundary condition.



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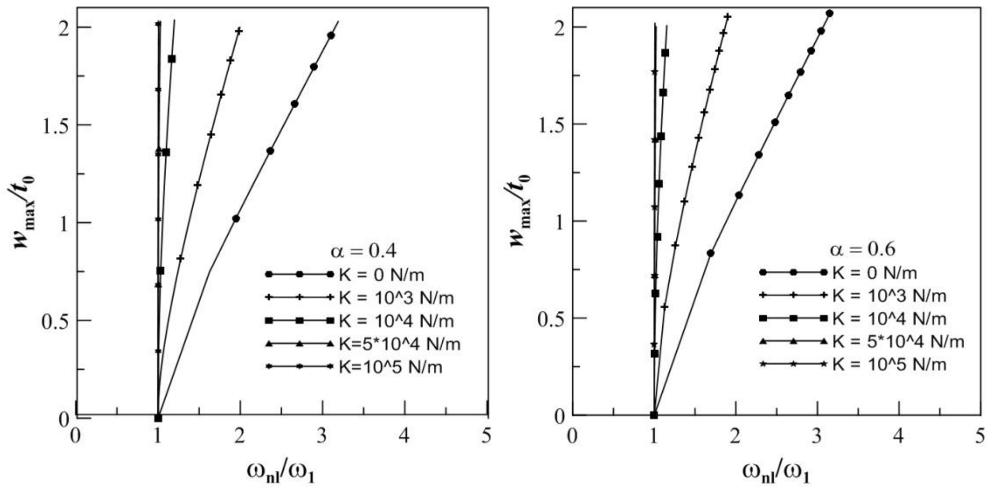

Figure 4. Backbone curve of AFG linear taper beam on elastic foundation with CF boundary condition.

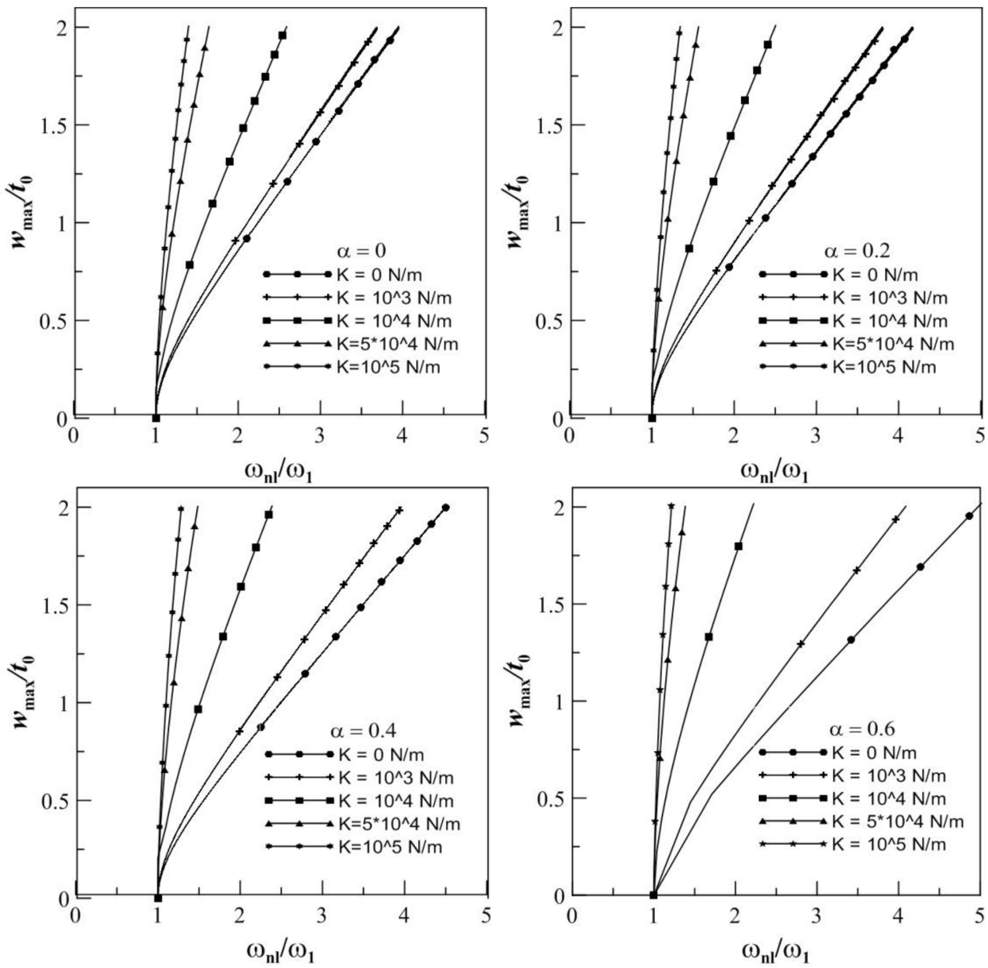

Figure 5. Backbone curve of AFG linear taper beam on elastic foundation with SS boundary condition.

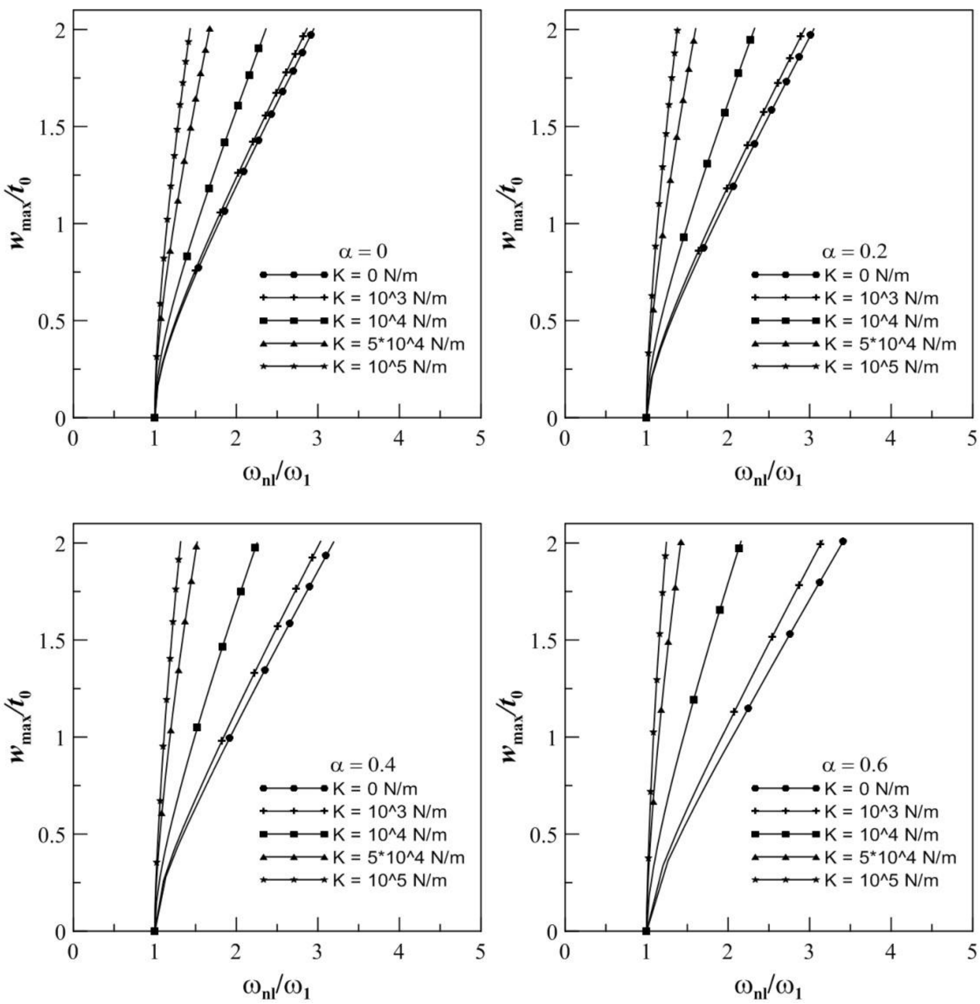

Figure 6. Backbone curve of AFG linear taper beam on elastic foundation with CS boundary condition.

The present analysis is based on a methodology where the solution of the static displacement field of the AFG beam on elastic foundation under uniformly distributed transverse loading is obtained followed by subsequent evaluation of the eigenvalues of the corresponding dynamic problem on the basis of converged static solution. The solution methodology of the static problem involves an iterative numerical scheme using successive relaxation due to presence of nonlinearity in the stiffness matrix. The number of Gauss points to be used for generation of results is taken as 24. The solution of the dynamic problem is obtained using Matlab subroutines. Following geometrical dimensions and material properties are used to generate the results: L = 1.0 m, b = 0.05 m, t_0 = 0.02 m, E_0 = 210 GPa, ρ_0 = 7850 kg/m^3.

Validation for the present formulation and solution technique is done by comparison with established results already available in literature. The backbone curve for fundamental mode of a clamped-clamped (CC) homogeneous uniform beam is compared with the results published by Gupta et al. [12] and the comparative plot is furnished in Figure 2. It can be seen from the figure that the matching of the two sets of results is satisfactory.

The fundamental frequencies for different parameters and different conditions are shown in Table 2. It is observed that for all the cases with the increase in taper parameter the natural frequency decreases. This decrement of frequency is due to the softening effect introduced by the decrease in cross-sectional area and moment of inertia. It is also observed that for all the cases with the increase in stiffness of the foundation the natural frequency increase as stiffer foundation makes the system more rigid.

It is well known that backbone curves of a vibratory system provide information about the relation of natural frequency and amplitude. In the present paper, large amplitude dynamic behaviour of the system is presented as backbone curves for the first mode in non-dimensional frequency amplitude plane, where the ordinate is dimensionless amplitude (w_{max}/t_0) and abscissa is normalized frequency (ω_{nl}/ω_l). In the present study (w_{max}/t_0) is taken as 2.0 for all cases. Figures 3-6 presents the backbone curves for axially FG tapered beams under uniformly distributed transverse loading for different combinations of taper patterns, stiffness variations, as well as boundary conditions. For all the cases, stiffness of the beam increases with increasing load due to geometric nonlinearity present in the system. This increased stiffness causes the increase in free vibration frequencies with increase in the deflection of the beam, as can be observed from any of the figures. So, hardening type nonlinear behaviour is exhibited by the system for all combinations of taper profile, stiffness values and boundary conditions.

4. Conclusions

In the present analysis, large amplitude responses of axially functionally graded slender taper beam with linear taper profile are investigated. The beam is further assumed to be on elastic foundation, modelled as a series of linear springs with specified spring stiffness. The beam is under the action of uniformly distributed transverse load, while four different flexural boundary conditions (CC, CS, SS and CF) are considered. However the present methodology can be applied for other type of classical and non-classical boundary as well. Also the methodology is flexible enough to account for other type of material gradation and taper pattern. Energy principle is applied for the mathematical formulation and the problem is solved in two parts, static and dynamic, respectively. First the static problem is solved for unknown static displacement fields and subsequently the dynamic problem is taken up based on those known displacement fields. For the static problem minimum total potential energy principle is utilized whereas for dynamic analysis the formulation is based on Hamilton's principle. The obtained results are validated from previously published results and were found to be in good agreement. Results pertaining to various boundary conditions, taper parameter and spring stiffness are furnished as backbone curve for the fundamental mode. For all combinations of the system parameters hardening type of nonlinearity is observed.

5. References

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