GRAPH THEORY AND ALGEBRAIC STRUCTURES
(MATH 2203)

Time Allotted : 3 hrs **Full Marks : 70**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 5 (five) from Group B to E, taking at least one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A
(Multiple Choice Type Questions)

1. Choose the correct alternative for the following: **10 × 1 = 10**

(i) In the additive group $(\mathbb{R}, +)$, $(9.1)^0$=
(a) 1 (b) 0 (c) -1 (d) 9.1.

(ii) The chromatic polynomial of the complete graph K_4 is
(a) $x^4 - x^3$ (b) $x(x-1)^3$
(c) $x^4 - 6x^3 + 11x^2 - 5x$ (d) $x^4 - 6x^3 + 11x^2 - 6x$

(iii) If a cyclic group G contains 8 distinct elements then the number of its generators is
(a) 1 (b) 2 (c) 3 (d) 8.

(iv) Which one of the following is not an integral domain?
(a) The set of all even numbers under addition and multiplication
(b) $\mathbb{R}$ under addition and multiplication
(c) $\mathbb{C}$ under addition and multiplication
(d) The set of all 2×2 matrices under matrix addition and multiplication.

(v) Which of the following group is not cyclic?
(a) $(\mathbb{Z}, +)$ (b) $(\{2n: n \in \mathbb{Z}\}, +)$
(c) $(\mathbb{Q}, +)$ (d) $(\mathbb{Z}_n, +)$

(vi) Which one of the following sets is closed under multiplication?
(a) $\{1, -1, 0, 2\}$ (b) $\{1, i\}$
(c) $\{1, \omega, \omega^2\}$ (d) $\{\omega, 1\}$.

(vii) Let G be a planar graph and G* be its dual. The number of vertices of G* is same as the
(a) number of vertices of G (b) number of edges of G
(c) number of regions of G (d) number of regions of G*

(viii) $\mathbb{Z}_6$ under addition and multiplication modulo 6 is
(a) a non-commutative ring (b) an integral domain
(c) a field (d) none of (a), (b), (c).

(ix) The inverse permutation of $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$ is
(a) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$
(c) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$.

(x) The number of elements in the symmetric group S_5 is
(a) 120 (b) 24 (c) 5 (d) 15.

Group – B

2. (a) Prove the following theorem: The chromatic number of C_n, a cycle with n vertices, is (i) 2 if n is even and (ii) 3 if n is odd.

(b) State Kuratowski's Theorem. Let e denote an edge of the complete bipartite graph $K_{6,6}$. Is $K_{6,6} - e$ a planar graph? Justify.

6 + 6 = 12

3. (a) Prove that the chromatic polynomial of a graph is a polynomial.

(b) In an examination seven subjects are to be scheduled: S_1, S_2, S_3, S_4, S_5, S_6, S_7. The following pairs of subjects have common students:
(S_1, S_2), (S_1, S_3), (S_1, S_4), (S_1, S_7), (S_2, S_3), (S_2, S_4), (S_2, S_5), (S_2, S_7), (S_3, S_4), (S_3, S_6), (S_3, S_7), (S_4, S_5), (S_4, S_6), (S_5, S_6), (S_5, S_7), (S_6, S_7)
How can the examination be scheduled so that no student has more than one examination on the same day. Show your work in detail.

5 + 7 = 12

Group – C

4. (a) Show that the set $G = \{(a + b\sqrt{2}) : a, b \in \mathbb{Q}\}$ is a group with respect to addition, where $\mathbb{Q}$ denotes the set of rationals.

(b) Show that if every element of a group $(G, *)$ be its own inverse, then it is an abelian group.

(c) Determine whether $*$ defined as follows in each of the following cases is a binary operation or not:
(i) On $\mathbb{Z}^+$, $a * b = a^b$ (ii) On $\mathbb{Q}$, $a * b = ab + 3$.

4 + 4 + (2 + 2) = 12

5. (a) Give an example of binary operation with suitable explanation for each of the following cases:
(i) non-associative (ii) non-commutative.

(b) Write down the multiplicative Cayley Table for U(8).

(c) Let G be group. If $a, b \in G$, then prove that $(ab)^{-1}=b^{-1}a^{-1}$.

(d) Give two examples of finite groups of order at least 4.

(2 + 2) + 3 + 3 + (1 + 1) = 12

Group – D

6. (a) Prove that the necessary and sufficient condition for a non-empty subset H of a group $(G, \circ)$ to be a subgroup is for all $a, b \in H$, $a \circ b^{-1} \in H$.

(b) Let H be a subgroup of a group G such that $[G:H]=2$. Then prove that H is a normal subgroup of G.

6 + 6 = 12

7. (a) State and prove Lagrange's Theorem regarding the order of a subgroup of a finite group.

(b) Prove that any two left cosets of H in a group G have the same cardinality.

6 + 6 = 12

Group – E

8. (a) Let d be an integer which is not a square. Prove that $\mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} : a, b \in \mathbb{Z}\}$ is an integral domain.

(b) Prove that a finite integral domain is a field.

5 + 7 = 12

9. (a) Determine the smallest subring of $\mathbb{Q}$ that contains $\frac{1}{2}$. (That is, find the subring S with the property that S contains $\frac{1}{2}$ and if T is any subring containing $\frac{1}{2}$ then T contains S.) Show your work.

(b) Prove that the set of Gaussian integers $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ is a subring of the set of all complex numbers $\mathbb{C}$ under addition and multiplication of complex numbers.
(i) Does $\mathbb{Z}[i]$ contain the multiplicative inverse of i ? Justify.
(ii) Does $\mathbb{Z}[i]$ contain the multiplicative inverse of $1 + i$? Justify.

6 + 6 = 12