B.TECH/ CSE /4TH SEM/ MATH 2201/2018
NUMBER THEORY AND ALGEBRAIC STRUCTURES
(MATH 2201)

Time Allotted : 3 hrs **Full Marks : 70**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 5 (five) from Group B to E, taking at least one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A
(Multiple Choice Type Questions)

1. Choose the correct alternative for the following: **10 × 1 = 10**

(i) Which of the following is **not** an associative binary operation?
(a) matrix addition (b) arithmetical addition
(c) matrix multiplication (d) arithmetical subtraction.

(ii) If x is an element of a group G and $O(x)$=5, then
(a) $O(x^{10}) = 5$ (b) $O(x^{15}) = 5$
(c) $O(x^{23}) = 5$ (d) $O(x^{20}) = 5$.

(iii) Define a relation on the set of integers $\mathbb{Z}$ such that aRb "iff " a – b is even. Then R is
(a) reflexive only (b) reflexive and symmetric only
(c) symmetric and transitive only (d) an equivalence relation.

(iv) If the cyclic group G contains 12 distinct elements then the number of generators of the group is
(a) 1 (b) 2 (c) 3 (d) 4.

(v) Which of the following permutations is a cycle?
(a) $\begin{pmatrix}1 & 2 & 3 & 4\\4 & 1 & 2 & 1\end{pmatrix}$ (b) $\begin{pmatrix}1 & 2 & 3 & 4\\2 & 1 & 4 & 3\end{pmatrix}$
(c) $\begin{pmatrix}1 & 2 & 3 & 4\\4 & 3 & 2 & 1\end{pmatrix}$ (d) $\begin{pmatrix}1 & 2 & 3 & 4\\4 & 3 & 2 & 3\end{pmatrix}$.

(vi) Index of a subgroup H of a group G is 5 and its order is 3. The order of the group G is
(a) 8 (b) 10 (c) 15 (d) 25.

(vii) Let $G = \mathbb{Z}$. The mapping $f: (G, +) \longrightarrow (G, +)$ defined by $f(x) = x + 2$ is
(a) homomorphism (b) epimorphism
(c) not homomorphism (d) isomorphism.

(viii) If a divides c and b divides c with $gcd(a,b) = 1$ then
(a) c divides ab (b) ab divides c
(c) gcd(a,c)=1 (d) gcd(a,c)=2.

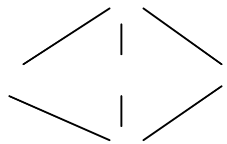
(ix) Which of the following is an example of non-commutative ring?
(a) Residue class ring modulo 6 (b) 2 × 2 matrices over a field
(c) The ring of Gaussian integers (d) All of above three.

(x) τ (1482)=
(a) 8 (b) 4 (c) 16 (d) 32.
(Where (n)= no. of positive divisors of n)

Group – B

2. (a) Solve the following system of linear congruence by using Chinese Remainder Theorem:
$$x \equiv 3(\text{mod } 7)$$
$$x \equiv 5(\text{mod } 9)$$
$$x \equiv 4(\text{mod } 5)$$

(b) Define distributive lattice. Give an example. Is the lattice with the following Hasse diagram distributive? Justify.



6 + (2 + 2 + 2) = 12

3. (a) Using theory of congruence, find the remainder when $2^{73} + 14^3$ is divided by 11.

(b) Let D_{30} be the set of all positive divisors of 30. Define a relation '≤' on D_{30} given by "x ≤ y if and only if x divides y", x,y∈ D_{30}. Prove that (D_{30}, ≤) is a poset. Draw the Hasse diagram of the poset.

6 + 6 = 12

Group – C

4. (a) Determine whether the set *P(X)* of all subsets of a non-empty set *X*, under * defined by $A * B = A \cup B$, $A, B \in P(X)$ forms a group.

(b) Let S be a set having n elements. How many distinct binary operations can be defined on S? Explain your answer.

(c) Show that the following four matrices form a group under matrix multiplication:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

4 + 4 + 4 = 12

5. (a) Show that the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ is even, while the permutation $\begin{pmatrix} 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 3 \end{pmatrix}$ is odd.

(b) Find the images of the elements 3and 4, if $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & - & - & 4 \end{pmatrix}$ be an odd permutation.

(c) If a is a permutation given by (1 2 3 4), then show that the set $\{a, a^2, a^3, a^4\}$ forms a cyclic group.

3 + 3 + 6 = 12

Group – D

6. (a) State and prove Lagrange's theorem regarding the order of a subgroup of a finite group.

(b) If a be an element of order n in a group and p be prime to n, then a^p is also of order n.

6 + 6 = 12

7. (a) Prove that two left cosets of H in a group G have the same cardinality.

(b) Prove that every group of order less than 6 is abelian.

6 + 6 = 12

Group – E

8. (a) Prove that $\mathbb{Z}_8$ is not a homomorphic image of $\mathbb{Z}_{15}$.

(b) Define unit in a ring R with unity. In the ring $(\mathbb{Z}_n, +;)$ prove that an element $\overline{m}$ is a unit implies $\gcd(m, n)=1$.

6 + 6= 12

9. (a) Let R be a commutative ring with unity of characteristic 3. Compute and simplify $(a+b)^9$ for $a, b \in R$.

(b) Prove that the set of matrices $\left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$ is is a subring of $M_2(\mathbb{Z})$.

(c) Prove that the cancellation law holds in a ring R iff R has no divisors of zero.

3 + 3 + 6= 12