2014

MATHEMATICS - 1
(MATH 1101)

Time Alloted : 3 Hours ***Full Marks : 70***

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 5 (five) from Group B to E, taking at least one from each group.

Candidates are required to give answer in their own words as far as practicable

GROUP - A
(Multiple Choice Type Questions)

1. Choose the correct alternative for the following : **[10×1=10]**

i) If A is a non-singular matrix of order 4, then the rank of the matrix A^{-1} is

(a) 4 (b) $\frac{1}{4}$

(c) 2 (d) $\frac{1}{2}$

ii) If A, B are two non-zero square matrices such that AB = 0, then

(a) A and B are non-singular

(b) A is singular

(c) B is singular

(d) A and B are singular

iii) The rank of the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ is

(a) 5 (b) 2

(c) 1 (d) 6

iv) The series $\sum_{n=1}^{\infty} \frac{n}{2n+1}$ is

(a) convergent

(b) divergent

(c) neigher convergent nor divergent

(d) none of these

v) Which of the following function obeys Rolle's theorem in $[0,\pi]$

(a) x (b) sin x

(c) cos x (d) tan x

vi) If $y = \frac{x^n}{x-1}$, then $(y_n)_0 =$

(a) $-(n!)$ (b) $n!$

(c) $(-1)^n n!$ (d) none of these

vii) If $y = 5x^{100} + 3$, then the 100th derivative of y is

(a) 100! (b) 0

(c) 5×100 (d) $5 \times 100!$

viii) If u + v = x and uv = y, then $\frac{\partial(x,y)}{\partial(u,v)} =$

(a) u − v (b) u v

(c) u + v (d) u / v

ix) If the vector $\vec{V} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$ is solenoidal, then the value of a is

(a) 2 (b) 4

(c) 3 (d) – 2

x) Let ϕ be a scalar point function, then $\left|\vec{\nabla}\times\left(\vec{\nabla}\phi\right)\right| =$

(a) 1 (b) 0

(c) 2 (d) none of these

GROUP - B

2. (a) Using elementary row operations find the rank of the following matrix :

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 8 & 6 \\ 3 & 6 & 6 & 3 \end{bmatrix}$$

(b) If A, B are orthogonal matrices of same order and $|A| + |B| = 0$, prove that (A+B) is a singular matrix.

6+6 = 12

3. (a) If $(I - A)(I + A)^{-1}$ is orthogonal, show that A is a skew symmetric matrix. (I is the identity matrix of same order as A.)

(b) If A, B are two n^{th} order square matrices and B is non-singular, prove that A and $B^{-1}AB$ have same eigen values. **6+6 = 12**

GROUP - C

4. (a) Apply Lagrange's Mean Value Theorem to prove that the chord on the parabola $y = x^2 + 2ax + b$ joining the points at $x = \alpha$ and $x = \beta$ is parallel to its tangent at the point $x = \frac{1}{2}(\alpha + \beta)$.

(b) In Cauchy's Mean Value Theorem, if $f(x) = e^x$ and $g(x) = e^{-x}$, show that θ is independent of both x and h and is equal to $\frac{1}{2}$. **7+5 = 12**

5. (a) Examine the convergence of the following infinite series.

$$\frac{1}{2^3} - \frac{1}{3^3}(1+2) + \frac{1}{4^3}(1+2+3) - \frac{1}{5^3}(1+2+3+4) + \ldots..$$

(b) Using Maclaurin's series show that

$$\sin x > x - \frac{1}{6}x^3 \text{ if } 0 < x < \frac{\pi}{2}.$$ **6+6 = 12**

Group - D

6. (a) Show that

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{for}(x,y) \neq (0,0) \\ 0 & \text{for}(x,y) = (0,0) \end{cases}$$

is not continuous at (0,0)

(b) If $u = \cos^{-1}\left\{\frac{x+y}{\sqrt{x}+\sqrt{y}}\right\}$ then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0.$$

(c) If $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$ find $\frac{\partial (u, v)}{\partial (x, y)}$.

3+5+4 = 12

7. (a) If u = sin ax + cos ax, show that

$$u_n = a^n\,[1 + (-1)^n \sin 2ax]^{1/2}$$

(b) If $f(x,y) = \dfrac{x^2 - xy}{x+y}$, $(x,y) \neq (0,0)$

$= 0, \qquad (x,y) = (0,0)$

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at the origin. 6+6 = 12

GROUP - E

8. (a) Prove that $2(n-1)a^2 I_n = \dfrac{x}{(x^2+a^2)^{n-1}} + (2n-3)\, I_{n-1}$

where $I_n = \displaystyle\int \frac{dx}{(x^2+a^2)^n}$, n being a positive integer greater than 1.

(b) Evaluate $\displaystyle\iint_R \frac{\sqrt{a^2b^2 - b^2x^2 - a^2y^2}}{\sqrt{a^2b^2 + b^2x^2 + a^2y^2}}\,dxdy$, the region of integration R being the positive quadrant of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

5+7 = 12

9. (a) If $\phi \equiv \phi\,(x, y, z, t)$, prove that .

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + \vec{\nabla}\phi \cdot \frac{d\vec{r}}{dt},$$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and x, y, z are differentiable functions of t.

(b) Evaluate $\iint_S \vec{F}.\hat{n}ds$ where $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ and S is the surface of the cube bounded by x = 0, x = 1, y = 0, y = 1, z = 0, z =1. 6+6 = 12
