B.TECH/ AEIE / CSE /7TH SEM/ MATH 4182/2017

LINEAR ALGEBRA
(MATH 4182)

Time Allotted : 3 hrs **Full Marks : 70**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and
any 5 (five) from Group B to E, taking at least one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A
(Multiple Choice Type Questions)

1. Choose the correct alternative for the following: **10 × 1 = 10**

(i) If M is a 3 x 3 matrix and is diagonalisable, then the number of linearly independent eigenvectors of M is
(a) one (b) two (c) three (d) four.

(ii) The product of the eigenvalues of an n × n matrix A is
(a) $(\det A)^2$ (b) 0 (c) det A (d) $(-1)^n$.

(iii) Which of the following is **not** a vector space?
(a) $(\mathbb{R}^n, +)$ over $(\mathbb{R}, +, \cdot)$ (b) $(\mathbb{R}, +)$ over $(\mathbb{Q}, +, \cdot)$
(c) $(\mathbb{R}, +)$ over $(\mathbb{Z}, +, \cdot)$ (d) $(M_{m\times n}, +)$ over $(\mathbb{R}, +, \cdot)$.

(iv) Let |S| denotes the number of elements of a finite set S. If B_1 and B_2 are two bases of a vector space V then
(a) $|B_1| > |B_2|$ (b) $|B_1| < |B_2|$
(c) $|B_1| \le |B_2|$ (d) $|B_1| = |B_2|$.

(v) Consider the following matrix
$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix}$$
Which of the following is true?
(a) The columns are linearly independent
(b) The matrix has determinant -1
(c) The matrix is not invertible
(d) None of the above.

(vi) Which of the following mapping is not linear ?
(a) $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x,y) = (x+1, y)$
(b) $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x,y) = (x+y, x)$
(c) $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x,y) = (2x-y, x)$
(d) $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x,y) = (2x, 3y)$.

(vii) Which of the following statement is true?
(a) If $T: V_1 \longrightarrow V_2$ is a linear map, then Rank T + Nullity T = Im T
(b) A maximal linearly independent subset of a vector space is a basis
(c) If a subset of a vector space contains the vector θ , i.e. the null vector, then the set is linearly dependent
(d) A subset of a linearly independent set may be linearly dependent.

(viii) Let f : V→W be a linear transformation and dim V= 4. Which of the following may be correct?
(a) Rank(f) = 4, Nullity (f)=1 (b) Rank(f) = 3, Nullity (f) =1
(c) Rank(f) = 2, Nullity (f)=2 (d) both (b) and (c).

(ix) The matrix representation of $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $T(x,y,z) = (3x-2y+z, x-3y-2z)$ relative to standard basis is given by
(a) $\begin{pmatrix} 3 & 1 & 1 \\ -2 & -3 & -2 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & -2 & 1 \\ -1 & -3 & -2 \end{pmatrix}$
(c) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

(x) The system of equations x + 2y = 5, 2x + 4y = 7 has
(a) unique solution (b) no solution
(c) infinite number of solutions (d) none of these

Group – B

2. (a) Prove that if λ is an r - fold eigen value of an n × n matrix A, 0 is an r - fold eigen value of matrix A - I_n. (where I_n denotes the identity matrix of order n) .

(b) Let $q(x, y) = x^2 + 6xy - 7y^2$. Find an orthogonal substitution that diagonalizes q.

4 + 8 = 12

3. (a) Find the singular values of the matrix $A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$

(b) Define generalized inverse of a matrix. Find the generalized inverse G of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$. Also find if $AGA = A$

3 + (2 + 4 + 3) = 12

Group – C

4. (a) Let $S=\{\alpha,\beta,\gamma\}, T=\{\alpha,\beta,\alpha+\beta,\beta+\gamma\}$ be subsets of a real vector space V. Show that $L(S)=L(T)$.

(b) Show that W is not a subspace of $\mathbb{R}^3$, where W consists of all those vectors whose length does not exceed 1, i.e., $W=\{(a,b,c): a^2+b^2+c^2\le 1\}$.

(c) Determine whether or not the vectors (1,-2,1), (2,1,-1), (7,-4,1) are linearly dependent.

6 + 3 + 3 = 12

5. (a) Prove that the set $S=\{(1,0,1),(0,1,1),(1,1,0)\}$ is a basis of $\mathbb{R}^3$.

(b) Describe the null spaces of these three matrices: $A=\begin{pmatrix}1 & 2\\ 3 & 8\end{pmatrix}$; $B=\begin{pmatrix}1 & 2\\ 3 & 8\\ 2 & 4\\ 6 & 16\end{pmatrix}$ and $C=\begin{pmatrix}1 & 2 & 2 & 4\\ 3 & 8 & 6 & 16\end{pmatrix}$.

6 + 6 = 12

Group – D

6. (a) If $A=\begin{pmatrix}1 & 0\\ 1 & 1\\ 1 & 2\end{pmatrix}$; $b=\begin{pmatrix}6\\ 0\\ 0\end{pmatrix}$, find $\hat{x}$, p and P. (Notations have their usual meaning).

(b) Find the parabola $C+Dt+Et^2$ that comes closest(least square error) to the values $b=(0,0,1,0,0)$ at times $t=-2,-1,0,1,2$.

6 + 6 = 12

7. (a) Compute the QR decomposition of the following matrix using Gram-Schmidt process

$$\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix}$$

(b) Prove that every matrix transforms its row space onto its column space.

7 + 5 = 12

Group – E

8. (a) For the following linear transformations $T:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ find a matrix A such that $T(x)=Ax\ \forall\ x\in\mathbb{R}^2$. $T\begin{pmatrix}1\\ 1\end{pmatrix}=\begin{pmatrix}1\\ -2\end{pmatrix}$; $T\begin{pmatrix}2\\ 3\end{pmatrix}=\begin{pmatrix}-2\\ 5\end{pmatrix}$.

(b) Let $A=\begin{pmatrix}1 & -3\\ 3 & 5\\ -1 & 7\end{pmatrix}$, $b=\begin{pmatrix}3\\ 2\\ -5\end{pmatrix}$, $c=\begin{pmatrix}3\\ 2\\ 5\end{pmatrix}$ and define a transformation $T:\mathbb{R}^2\longrightarrow\mathbb{R}^3$ by $T(x)=Ax$. Then

i) Find an x in $\mathbb{R}^2$ whose image under T is b.

ii) Is there more than one x whose image under T is b?

iii) Determine if c is in the range of the transformation T.

6 + 6 = 12

9. (a) The set $\{1,t,e^t,te^t\}$ is a basis of a vector space V of functions $f:\mathbb{R}\to\mathbb{R}$. Let D be the differential operator on V with respect to t. Find the matrix representation of D in the given basis.

(b) Let $T:\mathbb{R}^3\to\mathbb{R}^3$ be a linear mapping defined by

$$T(x,y,z)=(2x,4x-y,2x+3y-z).$$

Show that T is invertible and find a representation for T^{-1}.

6 + 6 = 12