GRAPH THEORY AND ALGEBRAIC STRUCTURES (MATH 2203)

Time Allotted : 3 hrs **Full Marks : 70**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 5 (five) from Group B to E, taking at least one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A
(Multiple Choice Type Questions)

1. Choose the correct alternative for the following: **10 × 1 = 10**

(i) The chromatic polynomial of K_3 is
(a) $\lambda(\lambda-1)(\lambda-2)$ (b) $\lambda(\lambda+1)(\lambda+2)$
(c) $\lambda(\lambda-1)$ (d) $\lambda(\lambda-1)^2$

(ii) A triangle is a planar graph and has regions
(a) 0 (b) 2 (c) 4 (d) 1.

(iii) A group contains 12 elements then the possible number of elements in a subgroup is
(a) 3 (b) 5 (c) 7 (d) 11.

(iv) In the group $\{1, -1, i, -i\}$ under multiplication, order of $-i$ is
(a) 0 (b) 2 (c) 4 (d) -4.

(v) The number of elements in the symmetric group S_5 is
(a) 5 (b) 20
(c) 120 (d) 5^5.

(vi) Let G be a cyclic group containing 10 elements with generator x. Then x^{10} is
(a) x (b) x^{-1}
(c) identity (d) x^9.

(vii) The left and right cosets of a subgroup H in a group G are identical if
(a) H is finite (b) G is abelian
(c) G is finite (d) none of these.

(viii) $\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix} =$

(a) $\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix}$ (b) $\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}$
(c) $\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}$ (d) none of these.

(ix) Travelling Salesman problem is associated with
(a) Euler's Circuit (b) Hamiltonian Circuit
(c) Regular Graph (d) Isomorphic Graph.

(x) If the set of residue class modulo n is an integral domain then n is
(a) an odd integer (b) an even integer
(c) 1 (d) a prime integer.

Group – B

2. (a) Define chromatic polynomial of a graph. Find the chromatic polynomial for a complete graph with n vertices.

(b) (i) Show that $x^5 - 3x^4 + 3x^3 - x^2$ is not a chromatic polynomial of a tree.

(ii) Find the chromatic polynomial of $K_{1,n}$.

(2 + 5) + (3 + 2) = 12

3. (a) State and prove Euler's formula.

(b) What is the maximum number of edges in a planar graph with 10 vertices?

(c) If a graph has 250 faces and 300 edges, can it be planar?

6 + 3 + 3 = 12

Group – C

4. (a) Let S be a set having n elements. How many different binary operations can be defined on S?

(b) (i) Let G = $\left\{\begin{pmatrix}a & 0\\ 0 & 0\end{pmatrix} \middle| a \in \mathbb{R} - \{0\}\right\}$. Determine whether <G, ∗> is a group or not, where ∗ is the usual matrix multiplication.

(ii) If G is a group such that $a^2 = e$ for every $a \in$ G, show that G is abelian.

4 + (4 + 4) = 12

5. (a) Let $A = \begin{pmatrix}1\,2\,3\,4\,5\\ 2\,3\,1\,5\,4\end{pmatrix}$ and $B = \begin{pmatrix}1\,2\,3\,4\,5\\ 1\,3\,4\,5\,2\end{pmatrix}$ be two permutations. Show that $AB \neq BA$.

(b) Find the order of the permutation $\begin{pmatrix}1\,2\,3\,4\\ 3\,4\,2\,1\end{pmatrix}$.

(c) Show that the set G of all ordered pairs (a, b) with $a \neq 0$, of real numbers a, b forms a group with operation '∘' defined by,

$$(a, b) \circ (c, d) = (ac, bc + d)$$

4 + 2 + 6 = 12

Group – D

6. (a) Prove that every subgroup of a cyclic group is cyclic.

(b) Let G be a group. If $a, b \in G$ such that $a^4 = e$, the identity element of G and $ab = ba^2$, prove that $a = e$.

6 + 6 = 12

7. (a) A subgroup H of a group G is normal if and only if $gHg^{-1} = H$ for all $g \in G$.

(b) Prove that every cyclic group is abelian. Is the converse true? Justify your answer.

6 + (3 + 3) = 12

Group – E

8. (a) (i) Show that the ring of matrices S=$\left\{\begin{pmatrix}2a & 0\\ 0 & 2b\end{pmatrix} \middle| a, b \in \mathbb{Z}\right\}$ contains divisors of zero and does not contain the unity.

(ii) Prove that $\mathbb{Z}[x]$, the ring of all polynomials with integer coefficients, is an integral domain.

(b) Prove that the ring of matrices R=$\left\{\begin{pmatrix}a & b\\ -b & a\end{pmatrix} \middle| a, b \in \mathbb{R}\right\}$ is a field.

(4 + 3) + 5 = 12

9. (a) Define characteristic of a ring R. Let R be a commutative ring with unity of characteristic 4. Compute and simplify $(a+b)^4$ for a,b in R.

(b) Define divisors of zero in a ring R.

(c) Solve the equation $x^2 + 2x + 4 = 0$ in Z_6.

(2 + 5)+ 2 + 3 = 12