

ALGEBRAIC STRUCTURES (MTH2201)

Time Allotted : 2½ hrs

Full Marks : 60

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 4 (four) from Group B to E, taking one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A

1. Answer any twelve:

12 × 1 = 12

Choose the correct alternative for the following

- (i) Consider the binary relation $R = \{(x, y), (x, z), (z, x), (z, y)\}$ on the set $\{x, y, z\}$. Which one of the following is TRUE?
 (a) R is symmetric but not antisymmetric
 (b) R is not symmetric but antisymmetric
 (c) R is both symmetric and antisymmetric
 (d) R is neither symmetric nor antisymmetric.
- (ii) If $\phi: x \rightarrow \{\text{the highest prime factor of } x\}$ and domain of ϕ is $\{12, 14, 17, 22\}$, then range set of ϕ is
 (a) $\{3, 2, 17, 11\}$ (b) $\{3, 7, 17, 2\}$
 (c) $\{6, 7, 17, 11\}$ (d) $\{3, 7, 17, 11\}$.
- (iii) In a Hasse diagram for a finite poset, an element x is called "maximal" if:
 (a) $x \leq y$ for every element y in the poset
 (b) there is no element y such that $x < y$
 (c) it is connected to the least element
 (d) it is the unique greatest element.
- (iv) In the additive group $(\mathbb{Z}, +)$, $2^{-3} =$
 (a) $\frac{1}{8}$ (b) 6 (c) -6 (d) $-\frac{1}{8}$.
- (v) Which of the following groups is non-abelian?
 (a) $(\mathbb{R}, +)$
 (b) $(\mathbb{C}, +)$
 (c) Set of all n th order matrices w.r.t matrix addition
 (d) Set of all n th order non-singular matrices w.r.t matrix multiplication.
- (vi) If G is a cyclic group of order 8 with generator x , then
 (a) x^4 is also a generator of G (b) x^5 is also a generator of G
 (c) x^6 is also a generator of G (d) x^2 is also a generator of G .

- (vii) Which of the following groups is not a cyclic group?
 (a) $(\mathbb{Z}, +)$ (b) $(\mathbb{Z}_n, +)$
 (c) $(\mathbb{R}, +)$ (d) $(\{2n: n \in \mathbb{Z}\}, +)$.
- (viii) Let ϕ be a function defined on $\mathbb{R}$ defined by $\phi(x) = \frac{x}{2} \forall x \in \mathbb{R}$, then which of the following is TRUE about ϕ ?
 (a) Not a homomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}, +)$
 (b) Monomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}, +)$ but not epimorphism
 (c) Epimorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}, +)$ but not monomorphism
 (d) Automorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}, +)$.
- (ix) In the field $(\mathbb{Q}, +, \times)$, which of the following is a unit?
 (a) 3.14 (b) π (c) e (d) 0.
- (x) Let ϕ be a group homomorphism from $\mathbb{Z}$ to $\mathbb{Z}_8$ defined by $\phi(n) = [n] \forall n \in \mathbb{Z}$. Then the inverse of $\phi(3)$ is
 (a) $[3]$ (b) 3 (c) $[5]$ (d) 5.

Fill in the blanks with the correct word

- (xi) Let X be the set of the alphabets present in HERITAGE. Consider the poset $(\mathbb{P}(X), \leq)$, where the relation defined on the set is the subset relation. The width of the poset is_____.
- (xii) If every element in a bounded lattice has at least one complement, the lattice is called a _____ lattice.
- (xiii) The number of odd permutations in the symmetric group of degree 4, i.e., S_4 is _____.
- (xiv) The number of homomorphisms from $\mathbb{Z}_4$ to $\mathbb{Z}_{12}$ is _____.
- (xv) Let $(G, *)$ and $(H, \circ)$ be two groups. A homomorphism $\varphi: G \rightarrow H$ is a _____ if and only if $\ker \varphi = \{e\}$.

Group - B

2. (a) Let $A = \{a, b, c\}$ and let R and S be relations on A whose matrices are $MR = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, MS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. Find $S \circ R$. Is the relation reflexive, symmetric, anti-symmetric, or transitive? Justify your answer. [(MTH2201.1, MTH2201.5)(Understand/LOCQ)]
- (b) Let $S = \mathbb{N} \times \mathbb{N}$. Define the relation $\sim$ on S where $a, b \sim (c, d)$ if and only if $a+d = b+c$. Prove that $\sim$ is an equivalence relation. [(MTH2201.1, MTH2201.5)(Evaluate/HOCQ)]
- 6 + 6 = 12**
3. (a) Draw the Hasse diagram representing the partial ordering $a, b \mid a$ divides b on the set $S = \{1, 2, 3, 4, 6, 8, 12\}$. Is the poset a lattice? Justify your answer. [(MTH2201.1, MTH2201.5)(Evaluate/HOCQ)]

- (b) Define a distributive lattice. Prove that a lattice $(L, \leq, \vee, \wedge)$ is distributive if and only if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \forall a, b, c \in L$.

[(MTH2201.1, MTH2201.5)(Analyse/IOCQ)]

$$5 + (2 + 5) = 12$$

Group - C

4. (a) Let G be a group such that for any $a \in G$, $a^2 = e$, where e is the identity element of G . Prove that G is a commutative group.

[(MTH2201.2, MTH2201.3, MTH2201.4, MTH2201.6)(Analyse/IOCQ)]

- (b) Let $G = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a(\neq 0), b(\neq 0) \in \mathbb{R}, a^2 + b^2 = 1 \right\}$. Prove that G is an abelian group with respect to matrix multiplication.

[(MTH2201.2, MTH2201.3, MTH2201.4, MTH2201.6)(Evaluate/HOCQ)]

$$6 + 6 = 12$$

5. (a) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix}$ be two permutations. Show that $\alpha\beta \neq \beta\alpha$.

[(MTH2201.2, MTH2201.3, MTH2201.4, MTH2201.6)(Remember/LOCQ)]

- (b) Find the order and inverse of the permutation: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 5 & 1 & 6 & 7 & 3 \end{pmatrix}$.

[(MTH2201.2, MTH2201.3, MTH2201.4, MTH2201.6)(Remember/LOCQ)]

- (c) Let A_n denote the alternating group of degree n , i.e., the group of even permutation of $\{1, 2, 3, \dots, n\}$. Prove that A_n has $\frac{n!}{2}$ elements, for $n > 1$.

[(MTH2201.2, MTH2201.3, MTH2201.4, MTH2201.6)(Apply/IOCQ)]

$$4 + 2 + 6 = 12$$

Group - D

6. (a) Show that every proper subgroup of a group of order 6 is cyclic.

[(MTH2201.2, MTH2201.3, MTH2201.4, MTH2201.6)(Apply/IOCQ)]

- (b) Prove that the set of matrices $H = \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \right\}$ is a normal subgroup of the group of all 2×2 non-singular matrices with respect to matrix multiplication.

[(MTH2201.2, MTH2201.3, MTH2201.4, MTH2201.6)(Evaluate/HOCQ)]

$$6 + 6 = 12$$

7. (a) If G is an abelian group, then determine that $H = \{a^2 : a \in G\}$ is a subgroup of G or not.

[(MTH2201.2, MTH2201.3, MTH2201.4, MTH2201.6)(Evaluate/HOCQ)]

- (b) Use Fermat's little theorem to find the remainder of 7^{100} when divided by 13.

[(MTH2201.2, MTH2201.3, MTH2201.4, MTH2201.6)(Apply/IOCQ)]

- (c) Consider the group $(\mathbb{Z}_{10}, +)$ and the normal subgroup $H = \{[0], [5]\}$. Then find the quotient group $\mathbb{Z}_{10}/H$.

[(MTH2201.2, MTH2201.3, MTH2201.4, MTH2201.6)(Apply/IOCQ)]

$$6 + 3 + 3 = 12$$

Group - E

8. (a) Let $\phi: S_3 \rightarrow S_3$ is defined by $\phi(x) = x^2$, where $x \in S_3$ and $(S_3, \circ)$ is the symmetric group of order 3. Is ϕ a homomorphism from $(S_3, \circ)$ to $(S_3, \circ)$? Justify your answer.

[(MTH2201.2, MTH2201.3, MTH2201.4)(Understand/IOCQ)]

- (b) Prove that the ring $(\mathbb{Z}_n, +, \cdot)$ is an integral domain if and only if n is prime.
 [(MTH2201.2, MTH2201.3, MTH2201.4)(Evaluate/HOCQ)]
6 + 6 = 12
9. (a) Let $G = (\mathbb{C}', \cdot)$, $G' = (\mathbb{R}^+, \cdot)$, be two groups, where $\mathbb{C}' = \mathbb{C} - \{0\}$, the set of all non-zero complex numbers. Show that the mapping $\varphi: G \rightarrow G'$ defined by $\varphi(z) = |z|$, $z \in \mathbb{C}'$ is a homomorphism. Determine $\text{Ker } \varphi$ and $\text{Im } \varphi$.
 [(MTH2201.2, MTH2201.3, MTH2201.4)(Analyse/IOCQ)]
- (b) If $(R, +, \cdot)$ is a ring such that $a^2 = a \ \forall a \in R$ then prove that $a + a = 0 \ \forall a \in R$.
 [(MTH2201.2, MTH2201.3, MTH2201.4)(Apply/IOCQ)]
6 + 6 = 12
-

Cognition Level	LOCQ	IOCQ	HOCQ
Percentage distribution	12.5	50	37.5