MATHEMATICAL METHODS
(MTH2001)

Time Allotted : 2½ hrs **Full Marks : 60**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 4 (four) from Group B to E, taking one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A

1. Answer any twelve: **12 × 1 = 12**

Choose the correct alternative for the following

(i) The regular singular point of $(1-x^2)y'' - 2xy' + n(n+1)y = 0$
(a) Only 1 (b) Only -1
(c) both 1 and -1 (d) 0

(ii) The Frobenius method is particularly useful for solving differential equations when:
(a) The equation has constant coefficients.
(b) The equation has variable coefficients with a regular singular point.
(c) The equation has a regular point.
(d) The equation is non-linear.

(iii) Legendre polynomials $P_n(x)$ are orthogonal in which domain
(a) [-1, 1] (b) [0, 1]
(c) (0, ∞) (d) (-∞, ∞)

(iv) Which of the following recurrence relations do Legendre polynomials satisfy?
(a) $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0.$
(b) $(2n+1)xP_n(x) - nP_{n-1}(x) = (n+1)P_{n+1}(x) = 0.$
(c) $(n+1)P_n(x) - P_{n+1}(x) + nP_{n-1}(x) = 0.$
(d) $(n+1)P_{n+1}(x) - nP_{n-1}(x) = 0.$

(v) What is the order of the equation $xy^3\left(\frac{\partial y}{\partial x}\right)^2 + yx^2 + \frac{\partial y}{\partial x} = 0$
(a) Third order. (b) Second order.
(c) First Order. (d) Unidentified.

(vi) The wave equation in one dimension is given by:
(a) $u_t = c^2 u_{xx}$ (b) $u_{tt} = c^2 u_{xx}$
(c) $u_{tt} + c^2 u_{xx} = 0$ (d) $u_x = c^2 u_{tt}$

(vii) What is the differential equation satisfied by Hermite polynomials $H_n(x)$?
(a) $y'' - 2xy' + 2ny = 0$ (b) $y'' + 2xy' - 2ny = 0$
(c) $y'' - 2xy' + 2y = 0$ (d) $y'' + xy' - ny = 0$

(viii) Which of the following is a linear PDE?
(a) $u_t + uu_x = 0.$ (b) $u_t + xu_x = 0.$
(c) $u_t + u^2u_x = 0.$ (d) $u_t = \sin(u)\, u_x.$

(ix) The Laplace equation in two dimensions is:
(a) $u_{tt} - c^2u_{xx} = 0$ (b) $u_t + u_x = 0$
(c) $u_{xx} + u_{yy} = 0$ (d) $u_x + u_y = 0$

(x) The general solution of $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ is:
(a) $u(x,t) = f(x+ct) + g(x-ct)$ (b) $u(x,t) = f(x-ct)$
(c) $u(x,t) = f(x+t)$ (d) $u(x,t) = f(x)g(t)$

Fill in the blanks with the correct word

(xi) The Frobenius method is used to find solutions to linear differential equations near a _____ singular point.

(xii) The radius of convergence R of a power series is the distance from the center to the nearest _____ where the series fails to converge.

(xiii) The relationship between Bessel functions of the first kind $J_n(x)$ and the second kind $Y_n(x)$ is given by_____.

(xiv) The value of the Hermite polynomial $H_2(x)$ is _____.

(xv) Charpit's method is used to solve first-order non-linear partial differential equations of the form $F(x,y,z,p,q) = 0$, where p and q represent the partial derivatives of z with respect to _____ and _____.

Group - B

2. (a) Find the first four terms in each portion of the series solution around $x_0 = 0$ for the following differential equation $(x^2+1)y'' - 4xy' + 6y = 0$
[(MTH2001.1, MTH2001.2)(Evaluate/HOCQ)]

(b) Write the following as a series that starts at $n = 0$ instead of $n = 3$
$\sum_{n=3}^{\infty} n^2 a_{n-1}(x+4)^{n+2}$ *[(MTH2001.1, MTH2001.2)(Apply/IOCQ)]*

8 + 4 = 12

3. (a) Find the radius of convergence of the power series $\frac{x}{2} + \frac{1}{2}.\frac{3}{5}.x^2 + \frac{1}{2}.\frac{3}{5}.\frac{5}{8}x^3 + \cdots$
[(MTH2001.1, MTH2001.2)(Analyse/IOCQ)]

(b) Determine whether $x = 0$is an ordinary point or a regular singular point of the differential equation $2x^2\frac{d^2y}{dx^2} + 7x(x+1)\frac{dy}{dx} - 3y = 0$ (give proper explanation).
[(MTH2001.1, MTH2001.2)(Remember/LOCQ)]

(c) Find the indicial equation of $x^2y'' + (x+x^2)y' + (x-9)y = 0.$
[(MTH2001.1, MTH2001.2)(Apply/IOCQ)]

4 + 4 + 4 = 12

Group - C

4. (a) Define generating function for Bessel's function and hence show that $2nJ_n(x) = x\{J_{n-1}(x) + J_{n+1}(x)\}$ *[(MTH2001.3)(Remember/LOCQ)]*

   (b) Prove that $\int_{-1}^{1} \mathrm{P_m(x)P_n(x)dx} = 0$ for $\mathrm{m} \neq \mathrm{n}$. *[(MTH2001.3)(Analyse/IOCQ)]*

6 + 6 = 12

5. (a) Evaluate P_0, P_1, P_2 and P_3, where $P_n(x)$ denotes the Legendre polynomial of order n and then express $P(x) = 3P_3(x) + 2P_2(x) + 4P_1(x) + 5P_0(x)$. *[(MTH2001.3)(Remember/LOCQ)]*

   (b) Show that $\cos(x\cos\theta) = J_0 - 2J_2\cos(2\theta) + 2J_4\cos(4\theta) - \cdots$ *[(MTH2001.3)(Apply/IOCQ)]*

6 + 6 = 12

Group - D

6. (a) Apply Charpit's method to find a complete integral of $2xz - px^2 - 2qxy + pq = 0$ where $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$. *[(MTH2001.4, MTH2001.5)(Evaluate/HOCQ)]*

   (b) Obtain a partial differential equation by eliminating arbitrary constants from $z = ax + by + cxy$ where a, b, c are constants. *[(MTH2001.4, MTH2001.5)(Understand/LOCQ)]*

7 + 5 = 12

7. (a) Construct a partial differential equation by eliminating $f(x)$ and $g(y)$ from the relation $z = yf(x) + xg(y)$. *[(MTH2001.4, MTH2001.5)(Understand/LOCQ)]*

   (b) Solve $x(y^2 - z^2)p - y(z^2 + x^2)q = z(x^2 + y^2)$ where $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$. *[(MTH2001.4, MTH2001.5)(Apply/IOCQ)]*

6 + 6 = 12

Group - E

8. (a) Solve the following by the method of separation variables $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, where $u(x, 0) = 6e^{-3x}$; u is a function of x and t *[(MTH2001.5, MTH2001.6)(Apply/IOCQ)]*

   (b) Solve $(D^2 + 2DD' + D'^2)z = x\cos(y)$, where $D \equiv \frac{\partial}{\partial x}, D' \equiv \frac{\partial}{\partial y}$ *[(MTH2001.5, MTH2001.6)(Remember/LOCQ)]*

6 + 6 = 12

9. (a) A tightly stretched string with fixed end points $x = 0$ and $x = l$ is pulled from its middle portion and the initial position is given by $y = y_0 \sin^3\left(\frac{\pi x}{l}\right)$. It is released from rest at this position. Determine the displacement $y(x, t)$ of the string. *[(MTH2001.5, MTH2001.6) (Apply/IOCQ)]*

(b) Find the particular integral of the equation $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin(2x + 3y)$.

[(MTH2001.5, MTH2001.6)(Remember/LOCQ)]

8 + 4 = 12

Cognition Level	LOCQ	IOCQ	HOCQ
Percentage distribution	38.54	45.83	15.63