LINEAR ALGEBRA FOR DATA ANALYSIS
(MTH3131)

Time Allotted : 2½ hrs **Full Marks : 60**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 4 (four) from Group B to E, taking one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A

1. Answer any twelve: **12 × 1 = 12**

Choose the correct alternative for the following

(i) Given vectors $v_1 = (1,1)^T$, $v_2 = (1,0)^T$, the first orthonormal vector in the Gram-Schmidt process is:
(a) $\frac{1}{\sqrt{2}}(1,1)^T$ (b) $\frac{1}{2}(1,1)^T$
(c) $(1,1)^T$ (d) $\frac{1}{\sqrt{5}}(2,1)^T$.

(ii) Consider the quadratic form $q(x) = x^T \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} x$. Which statement is true?
(a) $q(x)$ is indefinite (b) $q(x)$ is positive definite.
(c) $q(x)$ is negative definite (d) $q(x)$ is semidefinite

(iii) A matrix A has singular values $\sigma = \{10, 8, 6, 1\}$. According to the Eckart-Young theorem, the Frobenius norm of the error, $||A - A_2||_F$ (where A_2 is the best rank-2 approximation of A) is
(a) 6 (b) $\sqrt{37}$ (c) $\sqrt{100}$ (d) 7.

(iv) Given the matrix $A = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$, its SVD is $U = I, \Sigma = A, V = I$. The best rank-1 approximation of A is
(a) $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$

(v) The L_1 norm of the vector $v = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ is
(a) $\sqrt{14}$ (b) 6 (c) 3 (d) 14.

(vi) Consider the quadratic form $(x) = \frac{1}{2}x^T AX - b^T x$, $A = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$. The value of $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ that minimizes this function is:
(a) $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(vii) Let $A(t) = \begin{bmatrix} t & 0 \\ 0 & 3t \end{bmatrix}$. The value of the derivative of its singular value $\sigma_2(t)$ is

(a) 1 (b) 3 (c) t (d) $3t$.

(viii) Given the matrices $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, what is the probability of selecting the second column-row pair in a randomized matrix multiplication using a non-uniform sampling strategy?

(a) 0.8 (b) 0.2 (c) 0.5 (d) 0.9.

(ix) The range of p for which the following payoff matrix is strictly determinable is

PLAYER B

PLAYER A

p	6	2
-1	p	-7
-2	4	p

(a) $p \geq -1$ (b) $p \leq 2$

(c) $-1 \leq p \leq 2$ (d) for any value of p.

(x) Consider the problem of minimizing the function $f(x) = x^2$using Gradient Descent. If the starting point is $x_0 = 3$ and the learning rate is $\alpha = 0.5$, the value of x_1 after the first iteration is

(a) 0 (b) 1.5 (c) 2 (d) 3.

Fill in the blanks with the correct word

(xi) If $5x_1^2 + 2x_1x_2 - x_2^2$ is a real quadratic form in two variables $x_1$and x_2, then the associated matrix is ___________________.

(xii) A game is said to have a saddle point if the maximin and minimax values of the game are _________.

(xiii) In the QR algorithm, a matrix is iteratively decomposed into an orthogonal matrix Q and an _____________ matrix R.

(xiv) In the SVD of a matrix $A = U\Sigma V$, the columns of U are the left singular vectors and are eigenvectors of ________.

(xv) The quadratic form $x^T Ax$ is positive definite if and only if the symmetric matrix A is a _______________ matrix.

Group - B

2. (a) Show that the quadratic form $x^2 + 2y^2 + 6z^2 - 2xz + 4yz$ is positive definite.

[(MTH3131.1, MTH3131.2)(Understand/LOCQ)]

(b) Use Gram – Schmidt process to the vectors $(1, 1, 0)$, $(0, 1, 1)$ and $(1, 0, 1)$ to obtain an orthogonal and corresponding orthonormal basis for $\mathbb{R}^3$ with the standard inner product.

[(MTH3131.1, MTH3131.2)(Evaluate/HOCQ)]

5 + 7 = 12

3. (a) Show that the matrix $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$ is diagonalizable and also find the diagonal form. *[(MTH3131.1, MTH3131.2)(Apply/IOCQ)]*

(b) Reduce the quadratic form $x^2 + 2y^2 + 3z^2 - 2xy + 4yz$ to the normal form and show that it is indefinite. *[(MTH3131.1, MTH3131.2)(Apply/IOCQ)]*

8 + 4 = 12

Group - C

4. Solve the overdetermined system $\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \approx \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ using QR factorization.

[(MTH3131.2, MTH3131.3, MTH3131.4)(Evaluate/HOCQ)]

12

5. Let $A = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 4 \end{pmatrix}$

(i) Compute the SVD of A.

(ii) Construct the best rank-1 approximation of A (in Frobenius norm) using its largest singular value and vectors.

(iii) Compute the approximation error $||A - A_1 \|_F$.

[(MTH3131.2, MTH3131.3, MTH3131.4)(Apply/IOCQ)]

(5 + 5 + 2) = 12

Group - D

6. (a) Let $A = \begin{bmatrix} 0 & 3 & 2 \\ 1 & 4 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Find the vector $x \in \mathbb{R}^3$ that satisfies $Ax = b$ and minimizes $||x||_2$. *[(MTH3131.5)(Analyse/IOCQ)]*

(b) Given the matrix $A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$, use the power method, starting with vector $x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, to numerically approximate the largest eigenvalue of A after 3 iterations.

[(MTH3131.5)(Evaluate/HOCQ)]

6 + 6 = 12

7. (a) Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Find the vector $x \in \mathbb{R}^3$ that satisfies $Ax = b$ and minimizes $||x||_2$. *[(MTH3131.5)(Analyse/IOCQ)]*

(b) Given matrices $A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, using a sampling of $s = 1$, compute the probability of selecting each column-row pair. Compute the approximate product and compare it to the exact result. *[(MTH3131.5)(Evaluate/HOCQ)]*

6 + 6 = 12

Group - E

8. (a) Use graphical method to reduce the following pay-off matrix to a 2×2 game and hence find the optimal strategies and the value of game:

PLAYER B

PLAYER A	2	-1	5	-2	6
	-2	4	-3	1	0

[(MTH3131.6)(Apply/IOCQ)]

(b) Perform three epochs of stochastic gradient descent to minimize the following functions:

$$f(x, y) = (x + 2y - 4)^2,$$

starting with the point (0,0) and learning rate $\alpha = 0.1$. *[(MTH3131.6)(Evaluate/HOCQ)]*

6 + 6 = 12

9. (a) Use graphical method to reduce the following pay-off matrix to a 2×2 game and hence find the optimal strategies and the value of game:

PLAYER B

PLAYER A	2	-1
	3	2
	-1	5
	-2	1

[(MATH4126.5)(Analyse/IOCQ)]

(b) Fit the model $y = mx + c$, to the dataset $(x, y) = \{(0,1), (1,3), (2,5)\}$, Use learning rate $\alpha = 0.15$ and initial parameters $(m, c) = (1,0)$. Perform three epochs of stochastic gradient descent. *[(MTH3131.6)(Evaluate/HOCQ)]*

6 + 6 = 12

Cognition Level	LOCQ	IOCQ	HOCQ
Percentage distribution	5.2	50	44.8