LINEAR ALGEBRA
(MTH3121)

Time Allotted : 2½ hrs **Full Marks : 60**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 4 (four) from Group B to E, taking one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A

1. Answer any twelve: **12 × 1 = 12**

Choose the correct alternative for the following

(i) If A be a singular matrix then an eigenvalue of A is
(a) 1 (b) 2
(c) -1 (d) 0.

(ii) The quadratic form $2x^2 + 3y^2 + 2z^2 - 2xy - 2yz$ is
(a) negative definite (b) indefinite
(c) positive definite (d) positive semi-definite.

(iii) A subset S of a vector space $V(F)$ forms a basis of $V(F)$ if S is linearly independent and
(a) $L(S) = S$ (b) $L(S) = V$
(c) $L(S) = F$ (d) $L(F) = V$.

(iv) The dimension of the vector space spanned by $(-3, 0, 1), (1, 2, 1)$ and $(3, 0, -1)$ is
(a) 1 (b) 2
(c) 3 (d) 0.

(v) The norm of $u = (-1, 2, 3)$ in $\mathbb{R}^3$ with standard inner product is
(a) 14 (b) -14
(c) $\sqrt{14}$ (d) -6.

(vi) If u and v be two vectors in an inner product space V, then which of the following statement is true?
(a) $\|u + v\| = \|u\| + \|v\|$ (b) $\|u + v\| \geq \|u\| + \|v\|$
(c) $\|u + v\| \leq \|u\| + \|v\|$ (d) both (b) and (c).

(vii) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ be a linear transformation and the dimension of $Ker\ T$ is 2. Then the dimension of $Im\ T$ is
(a) 1 (b) 2
(c) 3 (d) 4.

(viii) A linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by $T(x, y) = (x + y, x)\ \forall (x, y) \in \mathbb{R}^2$. The nullity of T is:

(a) 3 (b) 2
(c) 1 (d) 0.

(ix) For a linear transformation T, the dimension of $Ker\ T$ is called

(a) rank of T (b) nullity of T
(c) range of T (d) image of T.

(x) A linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by $T(x, y) = (x, 0)\ \forall (x, y) \in \mathbb{R}^2$. Then the range of T is

(a) the x-axis (b) the y-axis
(c) the origin (0,0) only (d) the entire plane $\mathbb{R}^2$.

Fill in the blanks with the correct word

(xi) The real quadratic form associated to the matrix $\begin{pmatrix} 5 & 1 \\ 1 & -1 \end{pmatrix}$ is ______________.

(xii) Any subset containing $n + 1$ vectors of an n-dimensional vector space is linearly __________.

(xiii) The value of x for which the set of vectors $\{(1,2,1), (x, 3,1), (2, x, 0)\}$ are linearly independent in $\mathbb{R}^3$ is ____________________.

(xiv) If the vectors $(1, k, -3)$ and $(2, -5,4)$ are orthogonal then real value of k is _________.

(xv) In a linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ if $T\{(1, 0), (0, 1)\} = \{(2, 7), (1, 3)\}$, then the matrix representation of T with respect to the standard basis of $\mathbb{R}^2$ is ___________.

Group - B

2. (a) Find the eigenvalues and the corresponding eigenvectors of the matrix given by $A = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$. Also, find the algebraic multiplicity and geometric multiplicity of each of the eigenvalues. *[(MATH4126.1, MATH4126.6)(Apply/IOCQ)]*

(b) Reduce the quadratic form $x^2 + 2y^2 + 3z^2 - 2xy + 4yz$ to the normal form and show that it is indefinite. *[(MATH4126.1, MATH4126.6)(Analyse/IOCQ)]*

6 + 6 = 12

3. (a) Find the singular value decomposition of the matrix $A = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}$. *[(MATH4126.1, MATH4126.6)(Evaluate/HOCQ)]*

(b) Show that the eigenvalues of an orthogonal matrix are of unit modulus. *[(MATH4126.1, MATH4126.6)(Analyse/IOCQ)]*

8 + 4 = 12

Group - C

4. (a) Determine whether the set of vectors $\{(-1, 3, -2), (2, 4, 1), (1, 1, 1)\}$ forms a basis of the vector space $\mathbb{R}^3$ or not. *[(MATH4126.2)(Understand/LOCQ)]*

(b) If possible, express $\alpha = \begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix}$ in $M_2(\mathbb{R})$, the set of all 2×2 matrices with real entries, as a linear combination of the vectors $u = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, $v = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ and $w = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$. *[(MATH4126.2)(Remember/LOCQ)]*

(c) Determine whether the subset $S = \{(x, 2y, 3x) : x, y \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$ or not. If yes, find a basis of S. What is your conclusion about the dimension of S? *[(MATH4126.2)(Apply/IOCQ)]*

3 + 3 + 6 = 12

5. (a) Let V be the vector space $\mathbb{R}^3$ over the field of all real numbers and $W_1 = \{(0, y, z) : y, z \in \mathbb{R}\}$ and $W_2 = \{(x, y, 0) : x, y \in \mathbb{R}\}$.
(i) Find $W_1 \cap W_2$. Is it a subspace of V? Justify your answer.
(ii) Find $W_1 \cup W_2$. Is it a subspace of V? Justify your answer.
[(MATH4126.2)(Understanding/HOCQ)]

(b) Find the conditions on (x, y, z) such that it belongs to the span of the vectors $(2, 1, 0), (1, -1, 2)$ and $(0, 3, -4)$. *[(MATH4126.2)(Apply/IOCQ)]*

(3 + 3) + 6 = 12

Group - D

6. (a) State and prove the Pythagoras theorem for norms of vectors in an inner product space. *[(MATH4126.3. MATH4126.4)(Remember/LOCQ)]*

(b) If u and v be two vectors in a real inner product space and $\|u\| = \|v\|$, then show that $\langle u + v, u - v \rangle = 0$. *[(MATH4126.3. MATH4126.4)(Analyse/IOCQ)]*

(c) Let $\langle u, v \rangle$ be the standard inner product on $\mathbb{R}^2$. Let $\alpha = (1,2), \beta = (-1,1)$. If γ is a vector such that $\langle \alpha, \gamma \rangle = -1$ and $\langle \beta, \gamma \rangle = 3$, then find the vector γ. *[(MATH4126.3. MATH4126.4)(Apply/IOCQ)]*

3 + 3 + 6 = 12

7. (a) If V be a vector space of all polynomials in x with inner product given by $\langle f, g \rangle = \int_0^1 f(x) g(x) dx$, where $f(x), g(x) \in V$. Between the functions x and x^3, which one is closer to x^2 on the interval $[0, 1]$? Justify your answer. *[(MATH4126.3. MATH4126.4)(Understand/LOCQ)]*

(b) Use Gram-Schmidt process to the vectors $(1, 2, 2), (1, -1, 2)$ and $(1, 0, 1)$ to obtain an orthogonal and corresponding orthonormal basis for $\mathbb{R}^3$ with the standard inner product. *[(MATH4126.3. MATH4126.4)(Evaluate /HOCQ)]*

6 + 6 = 12

Group - E

8. (a) A function $T: \mathbb{R}^4 \to \mathbb{R}^4$ is defined by $T(x, y, z, t) = (2x, 3y, 0,0) \forall\, (x, y, z, t) \in \mathbb{R}^4$.
(i) Show that T is a linear transformation.
(ii) Find $Ker\ T$ and dimension of $Ker\ T$.
(iii) Find the dimension of $Im\ T$. *[(MATH4126.5)(Apply/IOCQ)]*

(b) A linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T(x, y, z) = (3x - 2y + z, x - 3y - 2z)$, $\forall (x, y, z) \in \mathbb{R}^3$. Find the matrix representing the linear transformation

T relative to the ordered bases $\{(0,1,1),(1,0,1),(1,1,0)\}$ of $\mathbb{R}^3$ and $\{(1,0),(0,1)\}$ of $\mathbb{R}^2$.

[(MATH4126.5)(Evaluate/HOCQ)]

6 + 6 = 12

9. (a) Determine the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ which maps the basis vectors $\{(0,1,1),(1,0,1),(1,1,0)\}$ of $\mathbb{R}^3$ to the vectors $(1,1,1),(1,1,1)$ and $(1,1,1)$ respectively. Hence verify the Rank-Nullity theorem for T.

[(MATH4126.5)(Analyse/IOCQ)]

(b) If a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T(x,y) = (2x+3y, x-y)$, then find the matrix of T with respect to the ordered basis $\{(1,0),(1,1)\}$ of $\mathbb{R}^2$. Also find the matrix of T^{-1} with respect to the ordered basis $\{(1,0),(1,1)\}$ of $\mathbb{R}^2$.

[(MATH4126.5)(Evaluate/HOCQ)]

6 + 6 = 12

Cognition Level	LOCQ	IOCQ	HOCQ
Percentage distribution	15.63	51.04	33.33