LINEAR ALGEBRA
(MATH 4126)

Time Allotted : 2½ hrs **Full Marks : 60**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 4 (four) from Group B to E, taking one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A

1. Answer any twelve: **12 × 1 = 12**

Choose the correct alternative for the following

(i) The geometric multiplicity of the eigenvalue 0 of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ is

(a) 0 (b) 1
(c) 2 (d) 3.

(ii) The matrix $A = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ is

(a) diagonalizable (b) not diagonalizable
(c) symmetric (d) idempotent.

(iii) A set of vectors $\alpha_i, i = 1, 2, \dots, n$, will be called linearly independent vectors, if for c_i scalars, $i = 1, 2, \dots, n$, $\Sigma_{i=1}^{n}(c_i.\alpha_i) = 0$ implies that

(a) at least one $c_i = 0$ (b) at least one $c_i \neq 0$
(c) all $c_i = 0$ (d) all $\alpha_i = 0$.

(iv) A set of single non-zero vector is

(a) linearly dependent (b) linearly independent
(c) basis (d) subspace.

(v) The set $V = \{(x, y) \in \mathbb{R}^2 : xy \geq 0\}$ is

(a) a vector space over $\mathbb{R}^2$ (b) a vector space over $\mathbb{R}$
(c) not a vector space over $\mathbb{R}$ (d) none of the above.

(vi) Which of the following set is an orthogonal set of vectors?

(a) $\{(0, 3, 4), (4, 2, 3), (0, 0, 1)\}$ (b) $\{(0, 3, 4), (1, 0, 0), (0, 2, 1)\}$
(c) $\{(0, 3, 4), (0, -4, 3), (5, 0, 0)\}$ (d) all of the above.

(vii) Consider the vectors $u = (1, 1, 1)$ and $v = (1, 2, -3)$ in $\mathbb{R}^3$. The value of $\langle u, v \rangle$ is

(a) 0 (b) -1
(c) 2 (d) 14.

(viii) Consider the vectors $u = (1, 2, 1), v = (2, 1, -4)$ and $w = (3, 2, -1)$ in $\mathbb{R}^3$. Then which of the following statement is true?
(a) u is parallel to v
(b) v is parallel to w
(c) all of them are orthogonal to each other
(d) only u and v are orthogonal to each other.

(ix) If $T: \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation and $Ker\ T = \{\theta\}$, then
(a) T is not-one-to-one mapping
(b) T is one-to-one mapping
(c) T is onto mapping
(d) T is onto mapping but not one-to-one.

(x) Let V be a finite dimensional vector space and $T: V \to V$ be a linear mapping. Then which of the following is true?
(a) $\dim V =$ rank of T
(b) $\dim V = \dim(Im\ T)$
(c) $\dim V = \dim(Ker\ T)$
(d) $\dim V = \dim(Ker\ T) + \dim(Im\ T)$.

Fill in the blanks with the correct word

(xi) If $5x_1^2 + 2x_1x_2 - x_2^2$ is a real quadratic form in two variables $x_1$and x_2, then the associated matrix is ____________________.

(xii) The matrix A is orthogonally diagonalisable if and only if A is a ____________ matrix.

(xiii) The smallest subspace of a vector space V containing a set S is called the ____________________ of S.

(xiv) If u be a unit vector in an inner product space, then $\|u\|$ is _____________.

(xv) Let $T: \mathbb{R}^3 \to \mathbb{R}^4$ is a linear transformation and the dimension of $Im\ T$ be 2 then rank of T is _________.

Group - B

2. (a) A is a 3×3 real matrix having eigenvalues $2, 3$ and 1. $\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\1\\1\end{pmatrix}$ are the eigenvectors of A corresponding to the eigenvalues $2, 3$ and 1, respectively. Find the matrix A. *[(MATH4126.1, MATH4126.6)(Understand/LOCQ)]*

(b) Find the orthogonal matrix Q in the QR decomposition of $A = \begin{pmatrix}-1 & -1 & 1\\ 1 & 3 & 3\\ -1 & -1 & 5\end{pmatrix}$. *[(MATH4126.1, MATH4126.6)(Evaluate /HOCQ)]*

6 + 6 = 12

3. (a) Let $A = \begin{pmatrix}2 & -2 & 0\\ -2 & 1 & -2\\ 0 & -2 & 0\end{pmatrix}$. Find an orthogonal matrix P such that $D = P^{-1}AP$ is a diagonal matrix. *[(MATH4126.1, MATH4126.6)(Apply/IOCQ)]*

(b) For which values of λ the quadratic form $x^2 + \lambda(y^2 + z^2) + 2xy$ is positive definite? Justify your answer. *[(MATH4126.1, MATH4126.6)(Understand/LOCQ)]*

8 + 4 = 12

Group - C

4. (a) Determine whether the set of vectors $\{(2,-1,1),(2,0,3),(1,1,-2)\}$ forms a basis of the vector space $\mathbb{R}^3$ or not. *[(MATH4126.2)(Understand/LOCQ)]*

   (b) Find the values of k so that the vectors $(1,-1,2)$, $(0,k,3)$ and $(-1,2,3)$ are linearly independent. *[(MATH4126.2)(Remember/LOCQ)]*

   (c) Show that the subset $S=\{(x,y,z)\in\mathbb{R}^3: 3x-y+z=0\}$ of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$. Hence, find a basis and dimension of S.

   [(MATH4126.2)(Apply/IOCQ)]

   4 + 2 + 6 = 12

5. (a) Find the conditions on $a,b\in\mathbb{R}$ so that the set of vectors $S=\{(a,b,1),(b,1,a),(1,a,b)\}$ in $\mathbb{R}^3$ is linearly dependent.

   [(MATH4126.2)(Analyse/IOCQ)]

   (b) Examine if the set $T=\{f\in C[0,1]: f(0)=0, f(1)=0\}$ is a subspace of the vector space $C[0,1]$. *[(MATH4126.2)(Analyse/IOCQ)]*

   (c) Give an example to show that the union of two subspaces of a vector space need not be a vector space itself. *(MATH4126.2)(Understand/LOCQ)]*

   5 + 5 + 2 = 12

Group - D

6. (a) If V be a vector space of all polynomials in t with inner product given by $\langle f,g\rangle = \int_0^1 f(t)g(t)dt$, where $f(t),g(t)\in V$. Now for $f(t)=3t-5$ and $g(t)=t^2$ find (i) $\langle f,g\rangle$, (ii) $\|f\|$ and (iii) $\|g\|$. *[(MATH4126.3. MATH4126.4)(Remember/LOCQ)]*

   (b) Use Gram – Schmidt process to the vectors $(1,0,1)$, $(1,1,1)$ and $(1,3,4)$ to obtain an orthogonal and corresponding orthonormal basis for $\mathbb{R}^3$ with the standard inner product. *[(MATH4126.3. MATH4126.4)(Evaluate/HOCQ)]*

   6 + 6 = 12

7. (a) Let a and b be two vectors in $\mathbb{R}^n$ such that their lengths are $\|a\|=\|b\|=1$ and the inner product $\langle a,b\rangle = a^T b = -\frac{1}{2}$. Then determine the length $\|a-b\|$.

   [(MATH4126.3. MATH4126.4)(Apply/IOCQ)]

   (b) Let $u=\begin{bmatrix}1\\1\end{bmatrix}$, $v=\begin{bmatrix}1\\-1\end{bmatrix}$ and $V=Span\{u,v\}$. Do u and v form an orthonormal basis for V? If not, then find an orthonormal basis for V.

   [(MATH4126.3. MATH4126.4)(Understand/LOCQ)]

   (c) Prove that an orthogonal set of non-zero vectors in a Euclidean space is linearly independent. *[(MATH4126.3. MATH4126.4)(Analyse/IOCQ)]*

   3 + 3 + 6 = 12

Group - E

8. (a) A function $T:\mathbb{R}^3\to\mathbb{R}^3$ is defined by $T(x,y,z)=(x+y+z, 2x+y+2z, x+2y+z)$, $\forall(x,y,z)\in\mathbb{R}^3$. Show that T is a linear transformation. Find $Ker\ T$ and the dimension of $Ker\ T$. *[(MATH4126.5)(Apply/IOCQ)]*

(b) A linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is defined by $T(x, y, z) = (x + 2y + 3z, 2x + 3y + z, 3x + y + 2z), \ \forall(x, y, z) \in \mathbb{R}^3$. Find the matrix representing the linear transformation T relative to the ordered basis $\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$.

[(MATH4126.5)(Evaluate/HOCQ)]

6 + 6 = 12

9. (a) A function $T: \mathbb{R}^3 \rightarrow \mathbb{R}$ is defined by $T(x, y, z) = x + y + z \ \forall \ (x, y, z) \in \mathbb{R}^3$.

(i) Show that T is a linear transformation.

(ii) Find $Ker\ T$ and dimension of $Ker\ T$.

(iii) Find $Im\ T$ and dimension of $Im\ T$.

(iv) Verify the Rank-Nullity Theorem for T. *[(MATH4126.5)(Analyse/IOCQ)]*

(b) Determine whether the linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (2x + y - z, y + 4z, x - y + 3z), \forall(x, y, z) \in \mathbb{R}^3$ is invertible or not.

[(MATH4126.5)(Evaluate/HOCQ)]

8 + 4 = 12

Cognition Level	LOCQ	IOCQ	HOCQ
Percentage distribution	28.12	48.96	22.92