INTRODUCTION TO COMPLEX AND FOURIER ANALYSIS
(MTH2101)

Time Allotted : 2½ hrs **Full Marks : 60**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 4 (four) from Group B to E, taking one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A

1. Answer any twelve: **12 × 1 = 12**

Choose the correct alternative for the following

(i) The function $f(z) = |z|^2$ is
(a) nowhere continuous (b) continuous everywhere
(c) continuous for a certain region only (d) continuous only at origin.

(ii) If $z = r(\cos\theta + i\sin\theta)$, then $|z|^3$ is equal to
(a) $(\cos\theta + i\sin\theta)^3$ (b) $r^3(\cos\theta + i\sin\theta)^3$
(c) $\frac{r^3}{2}$ (d) r^3 .

(iii) The function $f(z) = \bar{z}$ is
(a) continuous at $z = 0$ (b) differentiable at $z = 0$
(c) analytic at $z = 0$ (d) differentiable everywhere.

(iv) The value of $\oint_C \frac{z^2+1}{z-2}dz$, C being $|z| = \frac{\pi}{3}$ is
(a) 0 (b) $2\pi i$
(c) πi (d) $10\pi i$.

(v) The value of $\oint_C \frac{2z^2+7z-1}{z+1}dz$ where C is $|z| = \frac{1}{2}$ is
(a) 0 (b) $3\pi i$
(c) $-\pi i$ (d) $\frac{\pi i}{2}$.

(vi) The period of the function $f(x) = \cos x + \cos\sqrt{8}x$ is
(a) $\pi + \frac{\pi}{2\sqrt{2}}$ (b) $2(1+\sqrt{2})\pi$
(c) $2\sqrt{2}\pi$ (d) 2π.

(vii) If the Fourier series of $f(x) = |\sin x|$ for $-\pi < x < \pi$, is given by $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ then the value of a_0 is
(a) $\frac{2}{\pi}$ (b) $\frac{4}{\pi}$
(c) π (d) 2π.

(viii) If $F\{f(x)\} = f(p), F\{G(x)\} = g(p)$ be the convolution of $F(x)$ and $G(x)$, then $F\{F(x) * G(x)\}$ is equal to:

(a) $f(p) * g(p)$ (b) $f(p)/g(p)$

(c) $f(p) + g(p)$ (d) $f(p) - g(p)$.

(ix) If $f(s)$ is the Fourier transform of $F(x)$, then the Fourier transform of $F(ax)$ is,

(a) $\frac{1}{a} f\left(\frac{s}{a}\right)$ (b) $af\left(\frac{s}{a}\right)$

(c) $\frac{1}{a} f(as)$ (d) $af(as)$.

(x) If the Fourier transform of, $F(s) = e^{-2|s|}$, then $f(x)$ is

(a) $\frac{2}{\pi(x^2+4)}$ (b) $\frac{4}{\pi(x^2+4)}$

(c) $\frac{-2}{\pi(x^2+4)}$ (d) $\frac{-4}{\pi(x^2+4)}$

Fill in the blanks with the correct word

(xi) If $u(x, y)$ is harmonic function, then $u_{xx} + u_{yy}$ is equal to ________________.

(xii) If C be the circle $|z| = 1$, then $\int_C \frac{\cos 2z}{z - \frac{\pi}{6}} dz$ is ________________.

(xiii) If $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ is the Fourier series of $f(x) = \frac{1}{4}(\pi - x)^2$, then a_0 is ________________.

(xiv) The period of $\cos 2\pi x$ is ________________.

(xv) If $0 < x < \infty$, then the Fourier sine transform of $f(x)$ is defined by ____________.

Group - B

2. (a) Prove that $u(x, y) = \frac{1}{2}\log(x^2 + y^2)$ is harmonic and find its conjugate harmonic function $v(x, y)$ so that $f = u + iv$ is analytic. *[(MTH2101.1)(Apply/IOCQ)]*

(b) Prove that the function $f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ is continuous everywhere but $f'(0)$ does not exist, although Cauchy-Riemann equations are satisfied at the origin. *[(MTH2101.1)(Remember/LOCQ)]*

6 + 6 = 12

3. (a) If $f(z) = |z|^2$, then show that $f(z)$ is not analytic at $z = 0$. *[(MTH2101.1)(Apply/IOCQ)]*

(b) Prove that $u(x, y) = y^3 - 3x^2y$ is harmonic and find its conjugate harmonic function $v(x, y)$ so that $f = u + iv$ is analytic. *[(MTH2101.1)(Remember/LOCQ)]*

6 + 6 = 12

Group - C

4. (a) Evaluate $\oint_C \frac{z^2+1}{z^2-1} dz$, where C is the circle

(i) $|z| = \frac{3}{2}$

(ii) $|z| = \frac{1}{2}$. *[(MTH2101.2)(Remember/LOCQ)]*

(b) Expand $f(z) = \frac{z}{(z+1)(z+3)}$ in a Laurent series valid for

(i) $1 < |z| < 3$

(ii) $|z| < 1$. *[(MTH2101.2)(Evaluate/HOCQ)]*

6 + 6 = 12

5. (a) Expand the function $f(z) = \frac{1}{(z^2+1)(z+2)}$ in the region

(i) $|z| < 1$

(ii) $1 < |z| < 2$. *[(MTH2101.2)(Evaluate/HOCQ)]*

(b) Evaluate $\int_C \frac{z+7}{z^2+2z+5} dz$, where $C: |z - i| = \frac{3}{2}$. *[(MTH2101.2)(Remember/LOCQ)]*

6 + 6 = 12

Group - D

6. (a) Find the Fourier series expansion of the function $f(x) = \begin{cases} \pi + 2x, & -\pi \le x < 0 \\ \pi - 2x, & 0 \le x \le \pi \end{cases}$.

Hence prove that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$. *[(MTH2101.3, MTH2101.4)(Evaluate/HOCQ)]*

(b) Find the half range Fourier sine series for the function $f(x) = e^x$ in the interval $0 < x < 1$. *[(MTH2101.3, MTH2101.4)(Remember/LOCQ)]*

6 + 6 = 12

7. (a) Find the Fourier series of the function $f(x) = |x|$ in $-1 \le x \le 1$.

Hence show that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$. *[(MTH2101.3, MTH2101.4)(Evaluate/HOCQ)]*

(b) Find the radius of convergence of the power series $x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \cdots$.

[(MTH2101.3, MTH2101.4)(Understand/LOCQ)]

7 + 5 = 12

Group - E

8. (a) Use Parseval's Identity to prove that $\int_0^\infty \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab\,(a+b)}$, $(a > 0, b > 0)$.

Hence show that $\int_0^\infty \frac{dt}{(1+t^2)^2} = \frac{\pi}{4}$. *[(MTH2101.5, MTH2101.6)(Evaluate/HOCQ)]*

(b) Find the Fourier cosine transform of $f(x) = \begin{cases} x, & 0 < x < 1, \\ 2 - x, & 1 \le x < 2, \\ 0, & x \ge 2. \end{cases}$

[(MTH2101.5, MTH2101.6)(Remember/LOCQ)]

7 + 5 = 12

9. (a) Find the Fourier sine transform of $f(x) = e^{-|x|}$. Hence evaluate $\int_0^\infty \frac{x \sin mx}{1+x^2} dx$.

[(MTH2101.5, MTH2101.6)(Evaluate/HOCQ)]

(b) Find the inverse Fourier transform of $F(s) = \frac{1}{s^2+1}$.

[(MTH2101.5, MTH2101.6)(Remember/LOCQ)]

7 + 5 = 12

Cognition Level	LOCQ	IOCQ	HOCQ
Percentage distribution	46.9	12.5	40.6