ALGEBRAIC STRUCTURES
(MTH2201)

Time Allotted : 2½ hrs **Full Marks : 60**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and any 4 (four) from Group B to E, taking one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A

1. Answer any twelve: **12 × 1 = 12**

Choose the correct alternative for the following

(i) The relation $R = \{(a,b): a,b \in \mathbb{Z}, ab > 0\}$ defined on $\mathbb{Z}$ is
(a) symmetric (b) reflexive and transitive
(c) symmetric and transitive (d) equivalence relation.

(ii) Let $X = \{2,5,9\}$, $Y = \{1,2,7\}$. The mapping $f: X \to Y$ defined as $f\{(2,1),(5,1),(9,2)\}$ is
(a) one – to – one (b) many to one
(c) onto (d) one – one.

(iii) Which of the following operations is not a binary operation?
(a) Usual addition over $\mathbb{Q}$ (b) Usual multiplication over $\mathbb{Q}$
(c) Usual division over $\mathbb{N}$ (d) Usual multiplication over $\mathbb{N}$.

(iv) In a group $(G,*)$ if $(x*y)^{-1} = x^{-1}*y^{-1}\ \forall\, x,y \in G$, then
(a) G is finite (b) G is infinite
(c) G is abelian (d) G is empty.

(v) The number of even permutations in the symmetric group of degree 5, i.e, $S_5$is:
(a) 60 (b) 36 (c) 100 (d) 120.

(vi) If G be a group of order 7, then G is necessarily a
(a) non-abelian group (b) cyclic group
(c) non-cyclic group (d) symmetric group.

(vii) In the group $(\mathbb{Z}_6, +)$, the order of $[4]$ is
(a) 4 (b) 3 (c) 2 (d) 1.

(viii) Which of the following is a subgroup of $(\mathbb{Z}, +)$?
(a) the set of all integers multiple of 3 (b) the set of all odd integers
(c) the set of all prime integers (d) $\{1, -1\}$.

(ix) Which of the following is an example of Integral Domain?
(a) $\mathbb{Z}_4$ (b) $\mathbb{Z}_6$ (c) $\mathbb{Z}_7$ (d) $\mathbb{Z}_{10}$.

(x) Which of the following is an example of non-commutative ring?
(a) residue class ring modulo 6 (b) 2×2 matrices over a field
(c) the ring of Gaussian integers (d) the ring of real numbers.

Fill in the blanks with the correct word

(xi) Consider the poset $\{1, 2, 3, 4, 6, 9\}$ with respect to 'divide' relation. The minimal element of the set is/are ________________.

(xii) The order of the permutation $(1\ 2\ 3\ 4)(5\ 6) \in S_6$ is ________________.

(xiii) The number of elements in the alternating group A_5 is ________________.

(xiv) The index of subgroup H of G is 5 and $O(H) = 3$. The order of the group G is _____.

(xv) The number of unit element(s) of the ring $\mathbb{Z}$ is ________________.

Group - B

2. (a) For two elements $a, b \in \mathbb{Z}^+$, $a \sim b$ if a^3 divides b^3. Prove that $(\mathbb{Z}^+, \sim)$ is a poset.
[(MTH2201.1, MTH2201.5)(Analyse/IOCQ)]

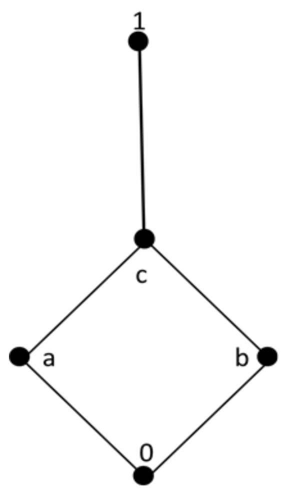
(b) Draw the Hasse diagram of the dual lattice of the lattice of divisors of 30 with respect to the divisibility relation. *[(MTH2201.1, MTH2201.5)(Create/HOCQ)]*

6 + 6 = 12

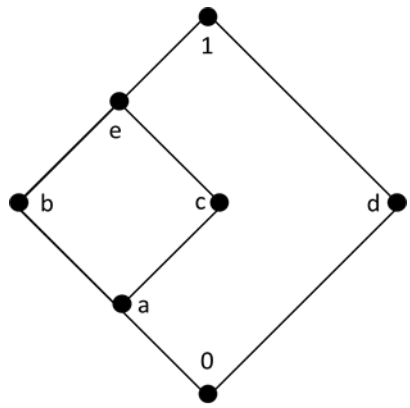
3. (a) Consider the poset $A = \{3, 5, 9, 15, 24, 45\}$ with the divisibility relation defined on A.
(i) Draw its Hasse diagram.
(ii) Find its maximum, minimum, maximal and minimal elements.
[(MTH2201.1, MTH2201.5)(Create/HOCQ)]

(b) Are the following lattices distributive or not? Explain.

(i)



(ii)



[(MTH2201.1, MTH2201.5)(Analyse/IOCQ)]

6 + 6 = 12

Group - C

4. (a) Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 1 & 3 & 6 & 4 \end{pmatrix}$ and $\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 5 & 3 & 4 \end{pmatrix}$. Compute the following: (i) $\sigma\gamma$ (ii) $\gamma\sigma$ (iii) γ^{-1}. *[(MTH2201.2,MTH2201.3,MTH2201.4,MTH2201.6)(Remember/LOCQ)]*

(b) Determine whether the following permutations in S_5 are even or odd. Find their orders. Justify your answers. (i) $(2\ 3\ 5)$, (ii) $(1\ 2)(2\ 3\ 4)(2\ 5\ 1)$, (iii) $(2\ 1\ 4\ 3)(3\ 5\ 1\ 2)$.

[(MTH2201.2, MTH2201.3, MTH2201.4, MTH2201.6)(Apply/IOCQ)]

6 + 6 = 12

5. (a) Let $(G,*)$ be a group. Prove that $(a*b)^{-1} = b^{-1}*a^{-1}\ \forall\, a,b \in G$.

[(MTH2201.2, MTH2201.3, MTH2201.4, MTH2201.6)(Analyse/IOCQ)]

(b) Prove that the inverse of an element of a group is unique.

[(MTH2201.2, MTH2201.3, MTH2201.4, MTH2201.6)(Understand/LOCQ)]

(c) Show that the set $G = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a group with respect to usual addition. Is this group abelian? Justify your answer.

[(MTH2201.2, MTH2201.3, MTH2201.4, MTH2201.6)(Evaluate/HOCQ)]

4 + 2 + 6 = 12

Group - D

6. (a) Let G be a group and $a \in G$ such that $O(a) = n$. Then prove that $a^m = e$ if and only if n divides m. *[(MTH2201.2, MTH2201.3, MTH2201.4, MTH2201.6)(Analyse/IOCQ)]*

(b) Show that every subgroup of a cyclic group is a normal subgroup.

[(MTH2201.2, MTH2201.3, MTH2201.4, MTH2201.6)(Analyse/IOCQ)]

(c) Find all the generators of the cyclic group $\mathbb{Z}_7$ with respect to the addition of residue classes modulo 7. *[(MTH2201.2, MTH2201.3, MTH2201.4, MTH2201.6)(Understand/LOCQ)]*

6 + 3 + 3 = 12

7. (a) State and prove Lagrange's theorem regarding the order of a subgroup of a finite group. *[(MTH2201.2, MTH2201.3, MTH2201.4, MTH2201.6)(Analyse/IOCQ)]*

(b) Consider the group $(\mathbb{Z}_5, +)$, i.e., the additive group of all integers modulo 5.
(i) Find the order of each of the elements of $\mathbb{Z}_5$.
(ii) Show that the group $(\mathbb{Z}_5, +)$ is a cyclic group.
(iii) Find all the generators of the cyclic group $(\mathbb{Z}_5, +)$.

[(MTH2201.2, MTH2201.3, MTH2201.4, MTH2201.6)(Apply/IOCQ)]

6 + 6 = 12

Group - E

8. (a) Let SO_2 be the special orthogonal matrices of order 2 and is given by

$$SO_2 = \left\{\begin{pmatrix} x & -y \\ y & x \end{pmatrix} : x, y \in \mathbb{R}, x^2 + y^2 = 1\right\}.$$

Define a map $\phi: \mathbb{R} \to SO_2$ as $\phi(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$. Prove that ϕ is a homomorphism from the additive group $(\mathbb{R}, +)$ to the multiplicative group $(SO_2, \times)$. *[(MTH2201.2, MTH2201.3, MTH2201.4)(Analyse/IOCQ)]*

(b) Prove that the only idempotent elements in an integral domain are zero and unity.

[(MTH2201.2, MTH2201.3, MTH2201.4)(Understand/LOCQ)]

6 + 6 = 12

9. (a) Let R be the ring of 2×2 matrices of the form $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$, where $a, b \in \mathbb{R}$. Show that although R is a ring that has no identity, we can find a subring S of R with an identity. *[(MTH2201.2, MTH2201.3, MTH2201.4)(Analyse/IOCQ)]*

(b) Let $(\mathrm{F}, +, \cdot)$ be a field and $a, b \in F$ with $b \neq 0$. Then show that $a = 1$ when $(ab)^2 = ab^2 + bab - b^2$. *[(MTH2201.2, MTH2201.3, MTH2201.4)(Evaluate/HOCQ)]*

6 + 6 = 12

Cognition Level	LOCQ	IOCQ	HOCQ
Percentage distribution	17.70	57.29	24