ADVANCED PROBABILITY AND INFORMATION THEORY
(MATH 3222)

Time Allotted : 2½ hrs **Full Marks : 60**

Figures out of the right margin indicate full marks.

Candidates are required to answer Group A and
any 4 (four) from Group B to E, taking one from each group.

Candidates are required to give answer in their own words as far as practicable.

Group – A

1. Answer any twelve: **12 × 1 = 12**

Choose the correct alternative for the following

(i) Let $X \in \{0,1\}$ and $Y \in \{0,1\}$ be two independent random variables. If $P(X=0)=p$ and $P(Y=0)=q$, then $P(X+Y \geq 1)$ is
(a) $pq+(1-p)(1-q)$ (b) pq (c) $p(1-q)$ (d) $1-pq$

(ii) Let X and Y be two random variables each taking three values $-1, 0, 1$ and having joint probability distribution

Y \ X	-1	0	1
-1	0	0	1/3
0	0	0	0
1	0	1/3	1/3

Then $E(X|Y=1)$ is
(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) $\frac{1}{3}$.

(iii) If the joint $p.d.f.$ is given by $f(x,y)=\begin{cases} cx(1-x), & 0 \leq x \leq y \leq 1 \\ 0, & otherwise \end{cases}$, then the value of c is
(a) 1 (b) 2 (c) 3 (d) 4

(iv) The measurements of spread or scatter of the individual values around the central point is called
(a) Measure of dispersion (b) Measure of central tendency
(c) Measure of skewness (d) Measure of kurtosis.

(v) Following is the transition probability matrix of a Markov chain:

$$\begin{matrix} & 1 & 2 & 3 \\ 1 & 1-2a & 2a & 0 \\ 2 & a & 1-2a & a \\ 3 & 0 & 2a & 1-2a \end{matrix}$$

then
(a) value of a is any real number (b) value of a is any positive real number
(c) value of a is less than 0.5 (d) value of a is $[0, 0.5]$

(vi) First order central moment for any distribution is
(a) 1 (b) -1 (c) 0 (d) 2.

(vii) The entropy of impossible event is
(a) 0 (b) 1 (c) 0.5 (d) -1

(viii) An event has two possible outcomes with probability $P_1 = 1/2$ and $P_2 = 1/64$. Calculate the rate of information with 16 outcomes per second is
(a) 24/3 (b) 64/6 (c) 38/4 (d) 1.

(ix) If X and Y are two independent and identically distributed random variables, then $Pr(X = Y)$ is
(a) $\geq 2^{-H(X)}$ (b) $\leq 2^{-H(X)}$ (c) $= 2^{-I(X,Y)}$ (d) $= -H(X)$.

(x) Let $A_{\epsilon}^{(n)}$ be a typical set with respect to $p(x)$ is the set of sequence $(x_1, x_2, \cdots, x_n) \in \chi^n$, then the maximum number of elements in the set $A_{\epsilon}^{(n)}$ is
(a) $2^{n(H(X)-\epsilon)}$ (b) $2^{n(H(X)+\epsilon)}$
(c) $2^{n(-H(X)-\epsilon)}$ (d) $2^{-n(H(X)-\epsilon)}$

Fill in the blanks with the correct word

(xi) If regression coefficients, $b_{xy} = -0.4$ and $b_{yx} = -0.9$, then correlation coefficient r_{xy} is __________ .

(xii) In Kurtosis, a frequency curve which looks more peaked than standard normal curve of bell shaped distribution is classified as _________________.

(xiii) If the moment generating function of a random variable X be $M_X(t)$, then the moment generating function of a random variable $Y = 2X - 5$ is _______.

(xiv) A fair coin is tossed 30 times. Let the random variable X be the number of heads that appear. The best upper bound on the probability of getting 20 or more heads is _____________.

(xv) The value of $H\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ is _______________________.

Group - B

2. (a) The length of time (in hours), in an "over 40" group of people to play one soccer match is normally distributed with a mean of 2 hours and a standard deviation of 0.5 hours. A sample of size $n = 50$ is drawn randomly from the population. Find the probability that the sample mean is between 1.8 hours and 2.3 hours. *[(MATH3222.1 & MATH3222.2)(Apply/IOCQ)]*

   (b) In the throwing of a pair of dice, let X and Y denotes the random variables denoting the number of sixes and the number of fives shown on the dice. Find the joint probability mass function of (X, Y) and the marginal probability mass function of X and Y. Also find $P(X + Y \geq 2)$. *[(MATH3222.1 & MATH3222.2) (Apply/IOCQ)]*

6 + 6 = 12

3. (a) Following is the joint probability distribution of X and Y:

X \ Y	1	2	3
0	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{10}$
2	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{5}$

(i) Is it a valid distribution? Give reason.
(ii) Find the marginal probability mass function of X and Y.
(iii) Are X and Y independent? Justify. *[(MATH3222.1 & MATH3222.2)(Apply/IOCQ)]*

   (b) A fair die is thrown 720 times. Use Chebyshev's inequality to obtain a lower bound for the probability of getting 91 to 149 sixes. *[(MATH3222.1 & MATH3222.2) (Apply/IOCQ)]*

6 + 6 = 12

Group - C

4. (a) The intelligence quotients of 55 students are as follows:

Class Mark	4	6	8	10	12	14	16	18	20	22
Frequency	4	5	8	12	10	6	4	3	2	1

Calculate the first three raw moments and hence the central moments. Find the measure of skewness for the intelligence quotient. Is it skewed to the left or right? *[(MATH3222.3 & MATH3222.4) (Analyze/IOCQ)]*

   (b) Consider the Markov chain with transition matrix P and state space $\{1, 2, 3\}$, where

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 \end{pmatrix}$$

Find its steady state solution. *[(MATH3222.3 & MATH3222.4) (Evaluate/HOCQ)]*

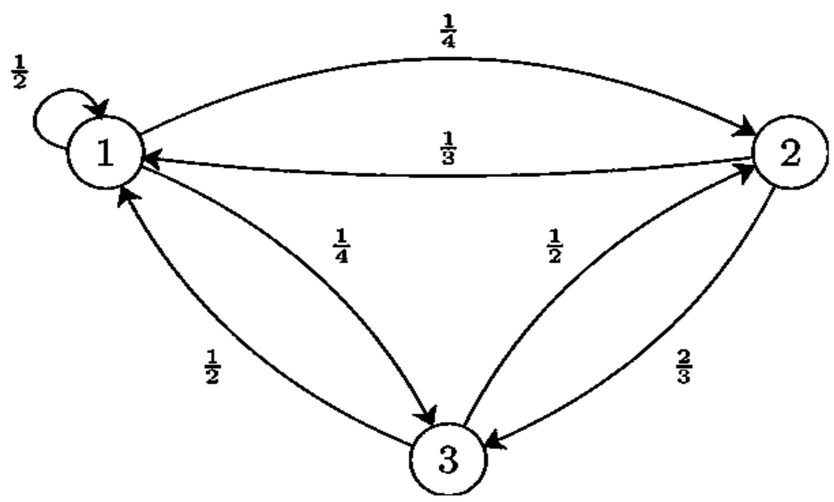
7 + 5 = 12

5. (a) Find a straight line $y = a + bx$ to fit given data, using the method of least squares:

x	0	1	2	3	4
y	1	1.8	3.3	4.5	6.3

[(MATH3222.3, MATH3222.4) (Analyze/IOCQ)]

   (b) Consider the Markov chain shown below:



(i) Is this chain irreducible? Justify your answer.
(ii) Is this chain aperiodic? Justify your answer.

(iv) Find the stationary distribution for this chain.
(v) Is the stationary distribution, a limiting distribution for the chain? Give reasons. *[(MATH3222.3, MATH3222.4)(Analyse/IOCQ)]*

5 + 7 = 12

Group - D

6. (a) A discrete source transmits message x_1, x_2 and x_3 with the probabilities 0.3, 0.4 and 0.3. The source is connected to the channel y_1, y_2 and y_3 by the following conditional probability matrix:

$$P(Y/X) = \begin{matrix} & y_1 & y_2 & y_3 \\ x_1 & 0.8 & 0.2 & 0 \\ x_2 & 0 & 1 & 0 \\ x_3 & 0 & 0.3 & 0.7 \end{matrix}$$

Calculate the mutual information between X and Y. *[(MATH3222.2 & MATH3222.5) (Understand/LOCQ)]*

(b) Use Jensen's inequality to find the appropriate inequalities between
(i) $E(X \log X)$ and $E(X) \log(E(X))$, $x \geq 0$
(ii) $E(\sin X)$ and $\sin(E(X))$, $0 \leq x \leq \pi$. *[(MATH3222.2 & MATH3222.5)(Apply/IOCQ)]*

6 + 6 = 12

7. (a) Consider that two sources S1 and S2 emit x_1, x_2, x_3 and y_1, y_2, y_3 with joint probability P(X, Y) as shown in the matrix. Calculate $H(X), H(Y)$ and $H(X, Y)$.

$$P(X,Y) = \begin{matrix} & x_1 & x_2 & x_3 \\ y_1 & 3/40 & 1/40 & 1/40 \\ y_2 & 1/20 & 3/20 & 1/20 \\ y_3 & 1/8 & 1/8 & 3/8 \end{matrix}$$

[(MATH3222.2, MATH3222.5)(Understand/LOCQ)]

(b) Let the random variable X have four possible outcomes $\{a, b, c, d\}$. Consider three distributions on this random variable

Symbols	$p(x)$	$q(x)$	$r(x)$
a	1/2	1/8	1/4
b	1/6	1/8	1/4
c	1/6	1/4	1/2
d	1/6	1/2	0

Calculate $D(p||q)$, $D(p||r)$ and $D(q||r)$. *[(MATH3222.2, MATH3222.5)(Remember/LOCQ)]*

6 + 6 = 12

Group - E

8. (a) Let

$$X = \begin{cases} 1, & \text{with probability } \frac{1}{3} \\ 2, & \text{with probability } \frac{1}{6} \\ 3, & \text{with probability } \frac{1}{2}. \end{cases}$$

Let $X_1, X_2, \cdots$ be drawn $i.i.d.$ according to this distribution. Find the limiting behaviour of $(X_1 X_2 \ldots X_n)^{\frac{1}{n}}$. *[(MATH3222.6)(Apply/IOCQ)]*

(b) Let the transition probability matrix of a two-state Markov chain $X \rightarrow Y \rightarrow Z$ is given as follows:

$$P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

Verify Data Processing lemma for the above problem. (i.e. $I(X; Z) \leq I(X; Y)$) *[(MATH3222.6)(Evaluate/HOCQ)]*

6 + 6 = 12

9. Let us consider the following Joint probability mass function on (X, Y)

$$P(X,Y) = \begin{matrix} & y_1 & y_2 & y_3 \\ x_1 & \frac{1}{14} & \frac{1}{7} & \frac{1}{14} \\ x_2 & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} \\ x_3 & \frac{1}{14} & \frac{1}{7} & \frac{1}{14} \end{matrix}$$

(i) Find the minimum probability of error estimator $\hat{X}(Y)$ and the associated probability.
(ii) Evaluate Fano's inequality for this problem and compare. *[(MATH3222.6)(Evaluate/HOCQ)]*

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Cognition Level	LOCQ	IOCQ	HOCQ
Percentage distribution	18.75	57.29	23.95